

Renormalization of NN Interaction with Chiral Two Pion Exchange Potential. Central Phases and the Deuteron.

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We analyze the renormalization of the NN interaction at low energies and the deuteron bound state through the Chiral Two Pion Exchange Potential assumed to be valid from zero to infinity. The short distance Van der Waals singularity structure of the potential as well as the requirement of orthogonality conditions on the wave functions determines that, after renormalization, in the 1S_0 singlet channel and in the $^3S_1 - ^3D_1$ triplet channel one can use the deuteron binding energy, the asymptotic D/S ratio and the S -wave scattering lengths as well as the chiral potential parameters as independent variables. We use then the asymptotic wave function normalization A_S of the deuteron and the singlet and triplet effective ranges to determine the chiral constants yielding c_1 , c_3 and c_4 . The role of finite cut-off corrections, the loss of predictive power due to uncertainties in the input data and the connection to OPE distorted waves perturbative approaches is also discussed.

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I. INTRODUCTION

The possibility suggested by Weinberg [1] and pioneered by Ray, Ordoñez and Van Kolck [2, 3, 4] of making model independent predictions for NN scattering using Effective Field Theory (EFT) methods and, more specifically, Chiral Perturbation Theory (ChPT) has triggered a lot of activity in recent years (for a review see e.g. Ref. [5]). In addition to the previous works, most subsequent calculations dealing with the specific consequences of ChPT have been focused in making predictions for NN scattering phase-shifts and deuteron properties based in the genuine Two Pion Exchange (TPE) chiral potentials [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27] although some incipient work has also recently been started implementing Three Pion Exchange effects [14, 15, 18, 28, 29]. In a given partial wave (coupled) channel with good total angular momentum the reduced NN potential ($U(r) = MV(r)$) in configuration space can schematically be written in local and energy independent form [7, 11, 12] (see Refs. [2, 3, 4] for an energy dependent representation) for any distances larger than a finite short distance radial regulator, r_c ,

$$U(r) = \frac{Mm^3}{f^2} W_{\text{LO}}(mr, g) + \frac{Mm^5}{f^4} W_{\text{NLO}}(mr, g, \bar{d}) + \frac{m^6}{f^4} W_{\text{NNLO}}(mr, g, \bar{c}_1, \bar{c}_3, \bar{c}_4) + \dots \quad (1)$$

where $W(x)$ are known dimensionless functions which are everywhere finite except for the origin where they exhibit power law divergencies, which demand the use of

some regularization. In writing the previous expression we have disregarded distributional contact terms (deltas and derivatives of deltas) which strengths are scheme dependent but do not contribute for $r \geq r_c > 0$. The potential is completely specified by the pion mass, m , the pion weak decay constant, f , the nucleon mass M , the axial coupling constant g , the Goldberger-Treiman discrepancy \bar{d}_{18} and three additional low energy constants $\bar{c}_1 = c_1 M$, $\bar{c}_3 = c_3 M$ and $\bar{c}_4 = c_4 M$ which can directly be deduced from the analysis of low energy πN scattering within ChPT [30, 31, 32, 33]. Given this information one can then solve the single or coupled channel Schrödinger equation imposing a regularity condition of the wave function at the origin for each separate channel. In this paper we will work under the assumption that the long range pieces of the potential should be iterated to all orders, but some perturbative analysis will also be done. Our motivation is to describe long range correlations between observables in the NN problem in a model independent way. As we will see below there is still some additional physical information required in the form of either counter-terms or short distance boundary conditions to make the problem well posed if one indeed wants to go to remove the regularization. They depend exclusively on the short distance behaviour of the potential through phases of the wave function. The number of phases depends crucially on the repulsive or attractive character of the potential. In this sense the power counting for the short distance interactions cannot be regarded as independent on the power counting of the singular chiral potentials. Non-perturbatively this materializes, after renormalization, in non-integer power counting for physical observables. This imposes severe limitations on the admissible structure of counterterms and the corresponding renormalization conditions of the quantum mechanical problem. Right away we hast to emphasize that *our approach is not the conventional EFT one of allow-*

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ing all possible short distance counterterms allowed by the symmetry, and to a certain extent our viewpoints are admittedly heterodox within the conventional EFT framework. However, based on the physical requirement of having small wave functions in the short range unknown region the basic and orthodox quantum mechanical requirements of completeness and orthogonality of states are deduced, providing a justification for the additional restrictions. We remind here the series of works by Phillips and Cohen [34, 35] (see also Re. [36]), where restrictions on zero range interactions were deduced for non-singular potentials based on the Wigner causality conditions. Here we extend their results also to the singular NN interactions of Eq. (1).

The theorem underlying the EFT developments is that if chiral symmetry is spontaneously broken down in QCD, the true NN potential at long distances is embedded in the parameter envelope of the general chiral NN potential, Eq. (1), and the chiral expansion provides a reliable hierarchy at those long distances. The hope is that compatible and perhaps accurate determinations of both πN and NN low energy data, bound states and resonances can be achieved with the same sets of parameters. The problem is that, in order to make truly model independent predictions, short distance ambiguities should be under control and their size smaller than the experimental data uncertainties used as input of the calculation. Only then can the renormalization program be carried out satisfactorily as it was done in the OPE case [37, 38, 39, 40, 41, 42], although, as recognized by Nogga, Timmermans and Van Kolck, this may be done at the expense of modifying the power counting [42] of the counterterms in favor of renormalizability (see also Ref. [41] for a complementary formulation in terms of boundary conditions). The present work analyses this problem extending our previous OPE renormalized calculations to NNLO TPE and its implication in the values of the chiral constants.

The determination of the chiral constants c_1 , c_3 and c_4 (in units of GeV^{-1} from now on) from πN scattering has been undertaken in several works and shows significant systematic discrepancies depending on the details of the analysis. In Heavy Baryon ChPT for low energy πN scattering [30] the values $c_1 = -1.23 \pm 0.16$, $c_3 = -5.94 \pm 0.09$ and $c_4 = 3.47 \pm 0.05$ were deduced with a sigma term of $\sigma(0) = 70\text{MeV}$. In Ref. [31] the analysis of low energy πN scattering inside the Mandelstam triangle yields $c_1 = -0.81 \pm 0.15$, $c_3 = -4.69 \pm 1.34$ and $c_4 = 3.40 \pm 0.04$ with, however, a bit too low sigma term $\sigma(0) = 40\text{MeV}$ as compared to ChPT. Unitarization methods reproducing the phase-shifts [32, 33] from threshold to the Δ resonance region conclude $c_1 = -0.43 \pm 0.04$, $c_3 = -3.10 \pm 0.05$ and $c_4 = 1.51 \pm 0.04$.

The values of the chiral constants c_1 , c_3 and c_4 also depend on regularization details of the NN chiral interaction. The πN values from Ref. [31] were taken in the nucleon-nucleon NNLO calculation of Ref. [10] with sharp and gaussian cut-offs $\Lambda = 0.6 - 0.8\text{GeV}$ in mo-

mentum space, and momentum dependent counter-terms were supplemented and determined from a fit to the NN data base based on the Partial Wave Analysis (PWA) of Ref. [43, 44]. Likewise, Ref. [19] constructs a NNLO chiral potential where channel dependent gaussian momentum space cut-offs in the range $\Lambda = 0.4 - 0.5\text{GeV}$ were used to fit the NN database [45]. The $N^3\text{LO}$ extension of this work [28] uses only one common cut-off and fixing $c_1 = -0.81$ produces $c_3 = -3.20$ and $c_4 = 5.40$. In Ref. [11] the NNLO calculation was done in configuration space with a short distance cut-off at $r = 1.4\text{fm}$ where an energy and channel dependent boundary condition was imposed and the fixed value $c_1 = -0.76 \pm 0.07$ was used to make a PWA to pp data yielding $c_3 = -5.08 \pm 0.24$ and $c_4 = 4.70 \pm 0.70$. An update of this calculation also including np data [21] generates $c_3 = -4.78 \pm 0.10$ and $c_4 = 3.96 \pm 0.22$. The calculations of Ref. [22, 23] improve the cut-off dependence of the potential in momentum space by using spectral regularization and take again gaussian cut-offs and fix $c_1 = -0.81$ yielding, after fitting the counter-terms to the NN PWA [43, 44], the values $c_3 = -3.40$ and $c_4 = 3.40$. The extension of this work to $N^3\text{LO}$ has been done in Ref. [29] keeping the same values for c_3 and c_4 and readjusting the counter-terms.

In a renormalized theory results should be insensitive to the auxiliary regularization method if the regulator is removed at the end. If a fit to the database proves successful, then the resulting parameters should be cut-off independent or at least the systematic uncertainty induced by the regularization should be smaller than the statistical errors induced by experimental data. Otherwise, the cut-off becomes a physical parameter. The first indication that finite cut-offs effects are sizeable in present calculations has to do with the variety of values that have been used in the literature for the low energy constants c_1 , c_3 and c_4 to adjust NN partial waves and deuteron properties [10, 11, 21, 22, 23, 28] (see also the comment in Ref.[24]). Obviously, we do not expect the values of the c 's to agree exactly, but the discrepancies should be at the level of the difference in the approximation¹. Since the data base is the same but the regularization schemes are different, one unavoidably suspects that these determinations of the LECS may perhaps be regularization and hence cut-off dependent.

To get a proper perspective on the issue of renormalization let us consider the size of the contributions of the chiral potential in configuration space at different distances. For instance, at $r = 1.4\text{fm}$ in the 1S_0 channel each order in the expansion is about an order of magnitude smaller than the preceding one. At short distances, however, the

¹ For instance, pion loops at NLO modify the contribution to c_3 by $\sim 3g_A^2 m^2 / (64\pi f^2) \sim 0.4/\text{GeV}$. This contribution must be taken into account when comparing numbers between [11, 21], [31] and the present approach. Only the extractions using the $N^3\text{LO}$ NN potential in Ref. [28] are made at the same order as those from [31].

situation is exactly the opposite, higher orders dominate over the lower orders. In the previous example of the 1S_0 channel, LO and NLO become comparable at $r \sim 0.9\text{fm}$, and NLO and NNLO become comparable at distances which value $r \sim 0.1 - 0.4\text{fm}$ depends strongly on the particular choice of low energy constants c_3 and c_4 . Actually, a general feature of the chiral NN potentials at NNLO has to do with their short distance behavior; they develop an attractive Van der Waals singularity $U \sim -MC_6/r^6$ similar to the one found for neutral atomic systems. In such a situation, the standard regularity condition at the origin only specifies the wave function uniquely if the potential is repulsive, but some additional information is required if the potential is attractive [46] (for a comprehensive review in the one channel case see e.g. Ref. [47]). Within the EFT framework the problem has been revisited in Ref. [48]. The net result is that the regularity condition at the origin tames the singularity [41] and, in fact, more singular potentials become less important at low energies.

In this work we reanalyze the NN chiral potential including TPE potential at NNLO. We carry out the analysis entirely in coordinate space following the ideas developed in our previous work [41] for the OPE potential. In configuration space the (renormalized) potential is finite except at the origin, a point which should carefully be handled, requiring a delicate numerical limiting procedure. For a singular potential in coordinate space, the corresponding potential in momentum space is not finite unless a short distance cut-off or a subtraction procedure at the level of the potential is implemented, hence modifying the potential everywhere and not just at high energies. This results generally in *two* cut-offs: one for the irreducible two point function and another for the Lippmann-Schwinger iteration [10, 19, 20, 22, 23, 28, 29]. The short distance coordinate space cut-off is unique and common both to the potential and the scattering solution.

Unlike previous works on the TPE potential we try to remove the cut-off completely taking the consequences seriously. This does not mean that finite cut-off calculations are necessarily incorrect or not entitled to describe all or some part of the data, but there are also good reasons for removing the cut-off and looking at the physical consequences. In the first place the limit exists in strict mathematical sense under well defined conditions, as the analysis below shows. This is a non-trivial fact, because calculations done in momentum space can only address this question numerically by adding counterterms suggested by an *a priori* power counting on the short distance potential. As shown in Ref. [42] this does not always work, and calculations may require some trial and error. Secondly, this is the only way we know how to get rid of short distance ambiguities, and thus to make truly model independent calculations. Third, the study of peripheral waves has proven to be successful by using perturbative renormalized amplitudes corresponding to irreducible TPE and iterated OPE where the cut-off has

been removed in the intermediate state [7, 20]. Peripheral waves mainly probe large distances in the Born approximation but they also see some of the short distance interaction due to re-scattering effects. Fourth, the advantage of renormalization is that one should obtain the same results provided one uses as input the same physical information, regardless whether the calculation is done in coordinate or momentum space, and also regardless on the particular regularization. Finally, a reliable estimate on the errors and convergence rate of the chiral expansion can be done, without any spurious cut-off contamination. In principle, the higher order in the chiral expansion the better, provided there is perfect errorless data to fit the increasing number of low energy constants appearing at any order. However, the chiral expansion may reach a limited predictive power because of finite experimental accuracy in the low energy constants used as input. The output inherits a propagated error which may eventually become larger than the experimental uncertainty². Finite cut-off uncertainties are not a substitute for propagating input experimental errors to the predictions of the theory, and can be regarded at best as a lower bound on systematic errors. In this paper we regard this possible cut-off dependence as purely numerical inaccuracies of the calculation, and not as a measure of the uncertainty in the predictions of the theory, so we make any effort to minimize these cut-off induced systematic errors.

In the process of eliminating the cut-off we find some surprises, and effects not explored up to now become manifest. Even for low energy scattering parameters and deuteron properties, where the description should be more reliable and robust we find systematic discrepancies in our calculation with values quoted in the literature and which we conclusively identify as finite cut-off effects. This might provide a natural explanation why calculations with different cut-off methods fitting the NN phase shifts [43, 44, 45] obtain different results for the chiral constants c_1 , c_3 and c_4 or why different values of the constants yield good fits to the data. According to our study, for the lowest phases the reason can partly be related to the dominance of short distance Van der Waals singularities for a system with unnaturally large scattering lengths or a weakly bound state as it is the case for the 1S_0 and $^3S_1 - ^3D_1$ channels. In some cases, they are as large as a 30% effect like in the effective range of the triplet 3S_1 channel. The size of the effect depends on the value of the low energy πN constants c_1 , c_3 and c_4 . Given the significant sensitivity of low energy NN properties and deuteron properties on these low energy πN constants we try to make a fit to some *low energy properties which uncertainties are reliably known and where we expect the chiral theory to be most reliable*. At this

² This issue has been illustrated in Refs. [49, 50, 51] for the case of $\pi\pi$ scattering at two loops, and will become clear in NN scattering below.

point we depart from the standard large scale fits to all phase shifts or partial wave analysis where the low energy threshold parameters are determined *a posteriori*. The assignment of statistical errors on the fitting parameters c_1 , c_3 and c_4 is often not addressed (see however Refs. [11, 21]) because the NN data bases used to fit the phase shifts [43, 44, 45] are treated as errorless. We also try to improve on this point within our framework.

The paper is organized as follows. In Sect. II we discuss the basic assumption of the smallness of the wave function in the short range unknown region and its consequences. We also analyze the constraints based on causality and analyticity of the S -matrix. In Sect. III we introduce the classification of boundary conditions which will be used along the paper to effectively renormalize the amplitudes both in the one-channel as well as the coupled channel case. We will also review the orthogonality constraints for singular potentials already used in our previous work [41] for the OPE potential. Sect. IV deals with the description of the singlet 1S_0 channel. From the superposition principle of boundary conditions we show how a universal form of a low energy theorem for the threshold parameters as well as for the phase-shift arises. In Sect. V we discuss the interesting triplet $^3S_1 - ^3D_1$ channel both for the deuteron bound state as well as the corresponding scattering states, where full use of the orthogonality constraints as well as the superposition principle of boundary conditions generates interesting analytical relations connecting deuteron and scattering properties. In Sect. VI a careful discussion of errors for our cut-off independent results is carried out. Also, a determination of the chiral constants based on low energy data and deuteron properties is made. In Sect. VII we present a simplified study on the significance of the chiral Van der Waals forces and the striking similarities with the full calculations for the s-waves. In Sect. VIII we show some puzzling results for the NLO calculation in the deuteron channel. We also comment the relation to finite cut-off calculations and the conflict between Weinberg counting and non-perturbative renormalization at NLO. We also outline possible solutions to this problem. In Sect. IX we analyze our results on the light of long distance perturbation theory, reinforcing the usefulness of non-perturbative renormalization due to an undesirable proliferation of counterterms. Finally, in Sect. X we summarize our conclusions.

For numerical calculations we take $f_\pi = 92.4\text{MeV}$, $m = 138.03\text{MeV}$, $M = M_p M_n / (M_p + M_n) = 938.918\text{MeV}$, $g_{\pi NN} = 13.083$ in the OPE piece to account for the Goldberger-Treiman discrepancy according to the Nijmegen phase shift analysis NN scattering [52] and $g_A = 1.26$ in the TPE piece of the potential. The values of the coefficients c_1 , c_3 and c_4 used along this paper are listed in Table I for completeness. The potentials in configuration space used in this paper are exactly those provided in Ref. [7, 11, 12] but disregarding relativistic corrections, $M/E \rightarrow 1$.

II. SHORT DISTANCE INSENSITIVITY CONDITIONS AND RENORMALIZATION

In this section we elaborate on the essential role played by standard quantum mechanical orthogonality and completeness properties of the wave functions in the rest of this paper. As we have already mentioned, our approach is unconventional from an EFT perspective, and at present it is unclear whether such properties have an EFT justification. At the same time one should say that many self-denominated EFT calculations do indeed normalize deuteron wave functions to unity and use energy independent regulators from which orthogonality relations follow automatically.

A. The inner and the outer regions

Similar to EFT, our basic assumption is that low energy physics should not depend on short distance fine details. This rather general principle can be made into a precise quantitative statement *in practice* for a quantum mechanical system. For the sake of clarity let us consider the singlet 1S_0 channel for positive energies. If we assume a short distance regulator r_c , above which our long distance (local) potential acts, the reduced Schrödinger equation in the outer region reads

$$-u''_{k,L}(r) + U_L(r)u_{k,L}(r) = k^2 u_{k,L}(r), \quad r > r_c \quad (2)$$

where the label L stands for long. Asymptotically behaves as

$$u_{k,L}(r) \rightarrow A \sin(kr + \delta(k, r_c)) \quad (3)$$

where A is an arbitrary normalization constant and the dependence on the short distance regulator r_c has been explicitly highlighted. In the inner region, the dynamics is *unknown* but we also expect it to be *irrelevant* provided $kr_c \ll 1$, i.e. if we assume the corresponding wavelength to be larger than the short distance scale. The potential can be deduced from perturbation theory in the full amplitude,

$$U(\vec{x}) = C_0 \delta(\vec{x}) + C_2 \{\nabla^2, \delta(\vec{x})\} + \dots + U_L(x) \quad (4)$$

where $U_L(x)$ corresponds to the expansion in Eq. (1) [7, 11, 12]. The distributional contact terms are regularization scheme dependent and correspond to polynomial terms in momentum space. Obviously, they do not contribute to the region $r > r_c$ for $r_c > 0$. The very nature of such a calculation already implies that C_0 , C_2 , etc. are perturbative corrections to the short range physics, but do not include possible non-perturbative effects. As will become clear below (Sect. III), finiteness of the physical phase shift in the limit $r_c \rightarrow 0$, implies a highly non-perturbative reinterpretation of the short range terms, even if the long range pieces are computed perturbatively.

Following [34, 36] it is useful to use a nonlocal and energy independent potential to describe the short distance dynamics

$$-u''_{k,S}(r) + \int_0^{r_c} U_S(r, r') u_{k,S}(r') dr' = k^2 u_{k,S}(r), \quad r < r_c \quad (5)$$

where the label S stands for short. This holds provided we are below any inelastic channel such as πNN . Above such threshold, any open channel should be included explicitly. The nonlocal short distance potential $U_S(r, r')$ encodes, in particular, contact terms (deltas and derivatives of deltas) which appear when the long distance potential $U_L(r)$ is computed in perturbation theory. These terms are in fact ambiguous (and hence unphysical) and depend on the regularization scheme used in the perturbative calculation, but are not essential since they do not contribute to the absorptive part and hence to the corresponding spectral function [7]. The important point is that these ambiguous distributional terms never contribute to the long range part provided $r_c > 0$ since in this case a compact support for distributional terms is guaranteed. This is a clear advantage of the radial cut-off we are using. In fact, our main motivation for carrying out the analysis in coordinate space is this clean separation between short and long distances. Many regulators, mainly those in momentum space, do not fulfill this condition, since the regulator effectively smears these short distance terms and intrudes somewhat into the long distance region³.

Finally, we need a matching condition connecting the inner and outer regions,

$$\frac{u'_{k,L}(r_c)}{u_{k,L}(r_c)} = \frac{u'_{k,S}(r_c)}{u_{k,S}(r_c)} \quad (6)$$

Viewed from the outer region this relation corresponds to an energy dependent boundary condition at a given short distance cut-off radius, r_c . Because of elastic unitarity we expect the state to be normalized, so that if we use a box of size a as an infrared regulator, we have

$$\int_0^{r_c} u_{k,S}(r)^2 dr + \int_{r_c}^a u_{k,L}(r)^2 dr = 1 \quad (7)$$

with a much larger than the range of the potential. This equation gives a quantitative separation between long distance and known physics and short distance and unknown physics. Obviously, any effective description based on the long range part should fulfill

$$\int_0^{r_c} u_{k,S}(r)^2 dr \ll 1 \quad (8)$$

which corresponds to the requirement of a *small wave function in the inner unknown region*. This is our basic condition from which most of our results follow. It should be realized that here long and short distances are intertwined through the matching condition, Eq. (6). In particular, an arbitrarily growing function at the origin cannot fulfill this condition even if value of the short distance cut-off is taken.

This kind of pathological situation actually occurs when dealing with the deuteron channel in the theory with no explicit pions [35], with OPE potential [41] and TPE potential at NLO in the Weinberg counting (see Sect. VIII and VIII C below). As an instructive and enlightening example, we illustrate the situation in Fig. 1 in the deuteron state for the quantity

$$P(r_c) = \int_0^{r_c} (u(r)^2 + w(r)^2) dr \quad (9)$$

(for notation see Sect. V) using the LO, NLO and NNLO potentials, Eq. (1), in the outer region, $r > r_c$, and matching to a free particle in the inner region $r < r_c$ (the precise form of the wave function inside turns out not to be essential)⁴. As one sees, there are cases such as a pure short distance theory where the asymptotic D/S ratio, η , and the deuteron binding energy are fixed to their experimental value. For $r_c < 1.4\text{fm}$ one has a minimum probability of about 20% in the inner region, and then $P(r_c)$ starts to grow. In this case the description cannot be considered effective below that critical value. A similar situation occurs in the NLO TPE case where one has a decreasing inside probability until one reaches $r_c \sim 0.8\text{fm}$ where it takes its minimum value, about 7% and then starts increasing. In contrast, for the LO (OPE) case where the deuteron binding energy is fixed to the experimental one and η is predicted [41] to have the value $\eta = 0.02633$ from the regularity condition of the wave function at the origin and also the NNLO (TPE) case where both numbers are fixed to experiment, the probability in the interior region is controlled and generally decreasing. This is a first and transparent illustration that the description based on any preconceived power counting is not necessarily consistent with the fact that short distance ambiguities are under control, since the wave function in the inner and unknown region does not become arbitrarily small as the cut-off is removed. Another possibility is to keep a finite cut-off, and we will analyze this point in more detail in Sect. VIII C.

As we will discuss in the next Sect. III, tight constraints on the structure of short distance counterterms must be fulfilled if the requirement $P(r_c) \rightarrow 0$ as $r_c \rightarrow 0^+$ is imposed.

³ For instance, the widely used gaussian regulator in momentum space [28, 29, 42] of a local potential in coordinate space corresponds indeed to a convolution of the original potential smeared over the region of size $1/\Lambda$.

⁴ In a momentum space formulation this is somewhat equivalent to cut-off the Lippmann-Schwinger equation above a given value Λ .

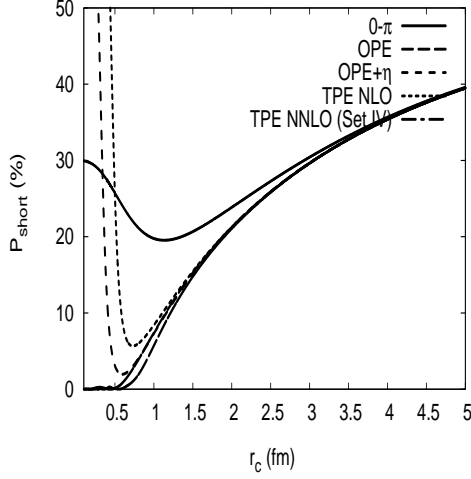


FIG. 1: Probability inside the short distance region as a function of the short distance cut-off radius r_c for the deuteron state for different approximations to the potential. In all cases the deuteron binding energy is fixed to the experimental value. $0-\pi$ means a pionless theory where in addition $\eta = 0.0256$ is also fixed to the experimental number. OPE means our previous work [41] where $\eta = 0.02633$ is deduced from the regularity condition of the wave function at the origin. $OPE + \eta$ corresponds to the OPE case when $\eta = 0.0256$ (the experimental value). TPE NLO and TPE NNLO (Set IV) correspond to the higher terms in Eq. (1) where also the experimental η value is taken.

B. Wigner bounds on the short distance contributions

To proceed further, we derive with respect to the energy Eq. (5) and one immediately gets after some algebra,

$$\frac{d}{dk^2} \left[\frac{u'_{k,S}(r_c)}{u_{k,S}(r_c)} \right] = - \frac{\int_0^{r_c} u_{k,S}(r)^2 dr}{u_{k,S}(r_c)^2} \leq 0 \quad (10)$$

whenever r_c is not a zero of the wave function.

The short range theory can be characterized by an *accumulated* phase shift $\delta_S(k, r_c)$, given by the solution of the truncated short range problem,

$$u_{k,S}(r) = \sin(kr + \delta_S(k, r_c)), \quad r > r_c \quad (11)$$

which fulfills, from Eq. (10),

$$\frac{d}{dk^2} [k \cot(kr_c + \delta_S(k, r_c))] \leq 0 \quad (12)$$

a condition equivalent to Wigner's causality condition [34]. Using an effective range expansion for the short distance phase shift

$$k \cot \delta_S = -\frac{1}{\alpha_{0,S}} + \frac{1}{2} r_{0,S} k^2 + \dots \quad (13)$$

we get the Wigner bound for the effective range

$$r_{0,S} \leq 2r_c \left[1 - \frac{r_c}{\alpha_{0,S}} + \frac{r_c^2}{3\alpha_{0,S}^2} \right] \quad (14)$$

where $\alpha_{0,S}$ and $r_{0,S}$ represent the scattering length and effective range when the short distance potential $U_S(r, r')$ is switched on from the origin up to the scale r_c or equivalently when the long distance potential is switched off from infinity down to r_c (see Refs. [39, 40] for more details). In a theory where the long distant potential is absent $U_L(r) = 0$, i.e., a pure short distance description, we have the obvious result that the short distance threshold parameters coincide with the physical parameters $\alpha_{0,S} = \alpha_0$ and $r_{0,S} = r_0$. Thus,

$$r_0 \leq 2r_c \left[1 - \frac{r_c}{\alpha_0} + \frac{r_c^2}{3\alpha_0^2} \right], \quad U_L(r) = 0 \quad (15)$$

which implies that $r_0 \leq 0$ for $r_c \rightarrow 0$. With the experimental values one gets the lowest short distance cut-off compatible with causality to be $r_c = 1.4$ fm. For a given long distance potential we just solve the equations from infinity inwards and look for the point where the Wigner condition is first violated. In Fig. 2 we plot the evolution of the Wigner bound on the effective range for the different approximations to the potential according to the expansion (1) as a function of the short distance cut-off radius. As we see, the lower bound on the radius is pushed towards the origin, and in fact *for the NNLO approximation there is no lower bound at all*. Thus, only for the NNLO TPE potential can one build the full strength of the experimental effective range without violation of the Wigner condition. We will see more on this in Sect. IV.

So far the discussion has been carried out for a fixed value of r_c . If we change the short distance radius, $r_c \rightarrow r_c + \Delta r_c$, we can use the matching condition, Eq. (6), to evaluate the change seen from the outer region. This results into a variable phase equation which has been analyzed extensively in our previous works [39, 40]. If we take the limit $r_c \rightarrow 0$, we get that the outer wave functions fulfills

$$\frac{d}{dk^2} \left[\frac{u'_k(0^+)}{u_k(0^+)} \right] = 0 \quad (16)$$

where the label L has been suppressed. Thus, the boundary condition becomes *energy independent* when the limit $r_c \rightarrow 0^+$ is taken if the inner wave function becomes arbitrarily small. Note that the limit is taken *from above* such that $r_c > 0$ ⁵. As we have said already, this justifies not considering contact terms in the potential. A direct consequence is that we get also that with the exception of the short distance scattering length $\alpha_{S,0}(0^+)$, which can be fixed by some renormalization condition

⁵ To illustrate this point, let us note that for potentials which do not diverge too strongly at the origin $r^2 U(r) \rightarrow 0$ such as OPE in the 1S_0 channel, there is some irreversibility in the process of integrating from *exactly* $r_c = 0$ out (which requires the regular solution, $u(0) = 0$) or integrating in towards the origin from above $r \rightarrow 0^+$ (which generally involves the irregular solution, $u(0) \neq 0$). See the discussion in Ref. [39] and in Sect. III C

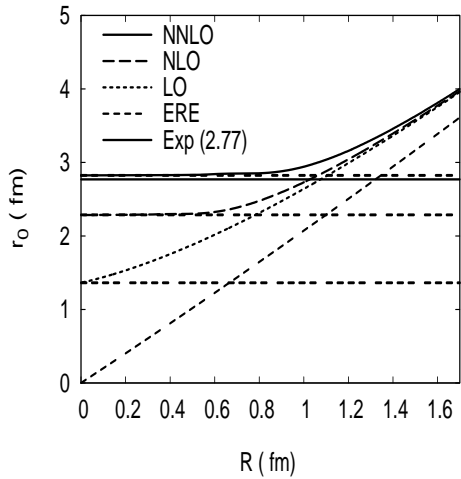


FIG. 2: Wigner bound on the effective range of the short distance interaction, (r.h.s. of Eq. (14)) as a function of the short distance cut-off radius for the contact interaction (ERE), the OPE interaction (LO), the TPE (NLO) and NNLO. We use the conditions $\alpha_0 = -23.74\text{fm}$ and $r_0 = 2.77\text{fm}$ when $r_c \gg 1/m_\pi$. Causality is violated when the curve crosses the experimental value. Straight horizontal lines correspond to the values obtained for the effective range when the short distance initial condition $r_{0,S}(0^+) = 0$ is assumed.

(see Sect. III), the remaining short distance threshold parameters are also zero in this limit,

$$r_{0,S}(0^+) = 0 \quad v_{2,S}(0^+) = 0, \dots \quad (17)$$

This energy independence of the boundary condition at the origin insures the orthogonality conditions between different energy states,

$$\int_{0^+}^{\infty} u_k(r)u_{k'}(r)dr = \delta(k - k'). \quad (18)$$

III. SHORT DISTANCE BEHAVIOR OF CHIRAL POTENTIALS, ORTHOGONALITY CONSTRAINTS AND THE NUMBER OF INDEPENDENT CONSTANTS

As we have said, chiral NN potentials, Eq. (1), although decay exponentially at large distances, become singular at short distances, where one has

$$U(r) = \frac{MC_n}{r^n} (1 + a_1 r + a_2 r^2 + \dots) \quad (19)$$

To avoid any misconception let us emphasize that the short distance behaviour of a long distance potential should be regarded as a long distance feature, i.e. a long wavelength property, since different long distance potentials yield different short distance behaviours. The short

distance properties of chiral potentials have nothing to do with short distance properties of the “true” potential, but renormalization and finiteness requires a very precise behaviour of the wave function when approaching the origin from long distances. In this section we classify the undetermined constants depending on the attractive or repulsive nature of the corresponding potentials in the single channel case and the eigenvalues of the potential matrix in the coupled channel case. In coordinate space and disregarding relativistic corrections the potentials in Eq. (1) are local and energy independent⁶. An important condition on the short distance behavior of the wave functions are the orthogonality constraints between states of different energy. For a regular energy independent potential these constraints are automatically satisfied, but for singular potentials they generate new relations relevant to the NN interaction. Our approach is not the conventional one of adding short distance counterterms following an *a priori* power counting on the long distance potential. Rather it is the power counting in the potential what uniquely determines the admissible form of the short distance physics if we want to reach a finite limit when the regulator is removed. This can only be achieved by choosing the *regular solution* at the origin, i.e. $u(0) = 0$. In Sect. VIII C we will show that irregular solutions generate divergent results after renormalization. Although this may look as a potential drawback of removing the cut-off it provides valuable insight on the form of the potential and the validity of the expansion (see Sect. VIII D).

A. One channel case

Let us first review the single channel case in a way that results for the coupled channel situation can be easily stated. The reduced Schrödinger equation for angular momentum l is

$$-u'' + U(r)u + \frac{l(l+1)}{r^2}u = k^2u. \quad (20)$$

For a power law singular potential at the origin of the form $U(r) = MC_n/r^n = \pm(R/r)^n/R^2$ with $n > 2$ and R

⁶ In momentum space and up to NNLO the long distance part of the potential depends on the momentum transfer q only and not on the total momentum k . Essential non-localities, i.e. contributions of the form $V(q, k) = L(q)k^2$ with $L(q)$ a non-polynomial function, depend weakly on the total momentum and appear first at N³LO [14, 15, 16, 17, 18, 20] due to relativistic $1/M^2$ one loop contributions. In coordinate space this weak non-locality corresponds to a modification of the kinetic energy term in the form of a general self-adjoint Sturm-Liouville operator, $-u''(r) \rightarrow -(p(r)u'(r))'$, with a singular $p(r)$ function at the origin and exponentially decaying at long distances. The present formalism can in principle be extended to include these features, and will be discussed elsewhere. Nevertheless, according to the results of Sect. VI (see Table IV) on the loss of predictive power already at NNLO there is a lack of phenomenological motivation.

the length scale dimension, the de Broglie wavelength is given by $1/k(r) = 1/\sqrt{|U(r)|}$ and the applicability condition for the WKB approximation reads $(1/k(r))' \ll 1$, so that for distances $r \ll R(n/2)^{2/(2+n)}$ one has a semi-classical wave function [46, 47, 48]. Keeping the leading short distance behavior one gets for attractive and repulsive singular potentials and any angular momentum the following behaviour for the *regular* solutions

$$u_A(r) \rightarrow C_A \left(\frac{r}{R}\right)^{n/4} \sin \left[\frac{2}{n-2} \left(\frac{R}{r}\right)^{\frac{n}{2}-1} + \varphi \right], \quad (21)$$

$$\text{for } U_A \rightarrow -\frac{1}{R^2} \left(\frac{R}{r}\right)^n \quad (22)$$

$$u_R(r) \rightarrow C_R \left(\frac{r}{R}\right)^{n/4} \exp \left[-\frac{2}{n-2} \left(\frac{R}{r}\right)^{\frac{n}{2}-1} \right] \quad (23)$$

$$\text{for } U_R \rightarrow +\frac{1}{R^2} \left(\frac{R}{r}\right)^n, \quad (24)$$

respectively. Here C_A and C_R are normalization constants and φ an arbitrary short distance phase. In the repulsive case we have discarded the irregular solution (a similar exponential with a positive sign) which would not allow to normalize states. For an attractive singular potential there is a short distance unknown parameter. This phase could, in principle, be energy dependent. Chiral potentials are, however, local and energy independent at NNLO at all distances, and become genuinely energy dependent at N³LO, due to relativistic $1/M^2$ corrections [14, 15, 16, 17, 18]⁷. Thus, if we require orthogonality of states with different energy (positive or negative) we get

$$\begin{aligned} 0 &= u'_k u_p - u_p u'_k \Big|_0 \\ &= \frac{1}{R} \sin(\varphi(k) - \varphi(p)). \end{aligned} \quad (25)$$

Hence, the phase φ is energy independent and could be fixed by matching the solution to the asymptotic large distance region (we assume a short range potential), e.g., by requiring a given value of the scattering length, α_l , at zero energy. In this way, a new and physical scale appears into the problem which is not specified by the potential. This is equivalent to the well known phenomenon of dimensional transmutation. Another possibility is to fix φ from a given bound state energy, $E = -B$. The new scale entering the problem is the corresponding wave number, $\gamma = \sqrt{MB}$. Note that although neither α_l nor γ can

be predicted from a singular potential, the orthogonality constraint does predict a correlation between them through the potential. Likewise, the phase shifts δ_l can be deduced from either α_l or γ by taking the same short distance phase φ . In the repulsive case there is no dimensional transmutation since the orthogonality condition follows from regularity at the origin, and the potential fully specifies the wave function. In this case, the scattering length and the spectrum are completely determined from the potential as for standard regular potentials.

B. Coupled channel case

We turn now to the two coupled channel case where the wave functions are denoted by a column vector (u, w) (for some particular cases see e.g. Refs. [41, 48, 53]). If we assume that at short distances the reduced potential behaves as

$$U \rightarrow M \frac{\mathbf{C}_n}{r^n} \quad (26)$$

where \mathbf{C}_n is a symmetric matrix of Van der Waals coefficients. Diagonalizing the matrix \mathbf{C}_n we get

$$\mathbf{C}_n = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} C_{n,+} & 0 \\ 0 & C_{n,-} \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \quad (27)$$

where $C_{n,\pm}$ are the corresponding eigenvalues and θ the mixing angle. Thus, at short distances we can decouple the equations to get

$$\begin{pmatrix} u \\ w \end{pmatrix} \rightarrow \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} u_+ \\ u_- \end{pmatrix} \quad (28)$$

where (u_+, u_-) are regular solutions as in the single channel case. So, in the two channel situation we have three possible cases depending upon the sign of the eigenvalues.

1. Both eigenvalues are negative, i.e., both eigenpotentials are attractive and $MC_{n,+} = -R_+^{n-2}$ and $MC_{n,-} = -R_-^{n-2}$ with R_{\pm} the corresponding scale dimension. In this case the short distance eigensolutions are oscillatory and there are two undetermined short distance phases, φ_+ and φ_- . Moreover for two states, (u_k, w_k) and (u_p, w_p) , with different energies we get the orthogonality constraint

$$\begin{aligned} 0 &= u'_k u_p - u_p u'_k + w'_k w_p - w_p w'_k \Big|_0 \\ &= \frac{1}{R_+} \sin(\varphi_+(k) - \varphi_+(p)) + \frac{1}{R_-} \sin(\varphi_-(k) - \varphi_-(p)). \end{aligned} \quad (29)$$

2. One eigenvalue is negative and the other is positive, $MC_{n,+} = R_+^{n-2}$ and $MC_{n,-} = -R_-^{n-2}$. One short distance eigensolution is a decreasing exponential and the other is oscillatory, so we have one

⁷ The subthreshold energy dependence from the virtual pion production channel $NN \rightarrow NN\pi$ which is in principle N³LO, disappears since in the heavy baryon limit the threshold $s_{\pi NN} = (2M + m)^2 = 4(M^2 + k^2)$ translates into a CM momentum $k = \sqrt{m(M + m/4)} \rightarrow \infty$.

short distance phase φ . In this case for two states (u_k, w_k) and (u_p, w_p) with different energies we get the orthogonality constraint

$$\begin{aligned} 0 &= u'_k u_p - u_p u'_k + w'_k w_p - w_p w'_k \Big|_0 \\ &= \frac{1}{R_+} \sin(\varphi_+(k) - \varphi_+(p)) . \end{aligned} \quad (30)$$

3. Both eigenvalues are positive, $MC_{n,+} = R_+^{n-2}$ and $MC_{n,-} = R_-^{n-2}$. Then, both short distance eigen-solutions are decreasing exponentials. There are no short distance phases. In this case the orthogonality relations are automatically satisfied.

This simple argument can be easily generalized to any number of coupled channels. The number of undetermined short distance phases corresponds to the number of attractive eigenpotentials at short distances. Orthogonality of the wave functions requires that all these short distance phases fulfill a generalized condition of the form of Eq. (29).

The orthogonality conditions require the determination of the short distance phases, as we did in Ref. [41] for the OPE case. This requires in general an improvement on the short distance behaviour to high orders. An alternative method is to impose the orthogonality constraints either in the single or coupled channel case by integrating in from infinity for a fixed energy, either positive or negative, and then impose the condition at a sufficiently short distance cut-off radius $r = r_c$. In the single channel case one would get the condition,

$$\frac{u'_k(r_c)}{u_k(r_c)} = \frac{u'_0(r_c)}{u_0(r_c)} , \quad (31)$$

if the zero energy state is taken as the reference state. An analogous relation holds for the coupled channel situation, namely

$$\begin{aligned} 0 &= u_k(r_c)u_0(r_c)' - u_k(r_c)'u_0(r_c) \\ &+ w_k(r_c)w_0(r_c)' - w_k(r_c)'w_0(r_c) . \end{aligned} \quad (32)$$

Obviously, in this procedure cut-off independence must be checked. For the TPE chiral potentials analyzed in this paper we find that $r_c = 0.1 - 0.2\text{fm}$ proves a sufficiently small value of the short distance cut-off.

C. Power Counting, counterterms and short distance parameters

As we see, the number of independent parameters is determined from the potential, although their value can be fixed arbitrarily, by some renormalization condition like, e.g., fixing scattering lengths to their physical value. This removes the cut-off in a way that short distances become less and less important. Now, if the potential is regular, i.e., $r^2|U(r)| \rightarrow 0$, one may *choose* between

the regular and irregular solution⁸. In the first case the scattering length is predicted while in the second case the scattering length becomes an input of the calculation. In either case the wave function is still normalizable at the origin. Singular potentials at the origin, i.e. fullfilling, $r^2|U(r)| \rightarrow \infty$, do not allow this choice if one insists on normalizability of the wave function at the origin. If the potential is repulsive, the scattering length is fixed while for an attractive potential the scattering length *must* be an input parameter. Furthermore, orthogonality of different energy solutions requires an energy independence of the boundary condition, so that in *all cases* the effective range, and higher order threshold parameters cannot be taken as independent parameters, in addition to the scattering lengths.

This can be translated into the language of counterterms quite straightforwardly. In momentum space, fixing α_0 arbitrarily corresponds to take a constant C_0 cut-off dependent and energy independent contribution to the potential $V_0(k', k)$ in the Lippmann-Schwinger equation. Likewise, fixing r_0 can be mapped as adding a term $C_2(k^2 + k'^2)$ to the potential. For higher coupled channel partial waves one fixes the scattering length $\alpha_{l,l'}$ one has instead terms of the form $C_{l,l'}k'^{l'}k^l$ in the potential $V_{l,l'}(k', k)$.

The OPE potential in the singlet 1S_0 is regular at the origin and hence one can take α_0 as an independent parameter or not (see Refs. [39, 40].) Actually, the smallness of the scattering length for the regular solution, suggests using the irregular solution. In the Weinberg's counting of the potential, at NLO one has TPE contributions in the potential. At short distances they behave as an attractive $1/r^5$ potential (see Sect. VIII), and then α_0 *must* be an independent parameter. At NNLO one has, again, a singular attractive $1/r^6$ potential (see Sect. IV), and thus an adjustable scattering length. This looks quite natural because increasing the order in the potential has a meaning and we can always compare the effect in the phase shifts of having a higher order potential with the same scattering length (See Sect. VIII). In this construction, if the next term in the expansion turned out to be more singular and repulsive the scattering length would be fully predicted from the potential.

The OPE potential in the triplet $^3S_1 - ^3D_1$ coupled channel corresponds to case 2) and hence one has *one* free parameter in addition to the OPE potential parameters. One may choose this parameter to be the deuteron binding energy (or alternatively the triplet S-wave scattering length). Any other bound state or scattering observables are predicted. This case was treated in great detail in our previous work [41]⁹. In the NLO TPE potential we have

⁸ Both cases comply to the normalizability condition at the origin, Eq. (8)

⁹ Relevant previous work on this channel was also presented in Ref. [38] and [40] where the orthogonality conditions were not

case 3) because both eigen potentials present a repulsive $1/r^5$ singularity (see Sect. VIII) and one would predict all observables from the potential parameters. Finally, in the NNLO TPE potential we have case 1) corresponding to an attractive-attractive (see Sect. V) and two additional parameters need to be specified for a state with a given energy. The orthogonality condition imposes a relation between two states of different energy, so that for all energies in the triplet channel one has *three* independent parameters. We will take these three parameters to be the deuteron binding energy, the asymptotic D/S ratio and the S -wave scattering length. The trend one observes when going from LO to NNLO is quite natural; as usual in ChPT one has more parameters at any order of the approximation. The NLO approximation poses, however, a problem since one seems to have more predictive power than at LO (See Sect. VIII for more details on the consequences of using our renormalization ideas literally for the conventional NLO potential).

For the NNLO TPE triplet ${}^3S_1 - {}^3D_1$ channel we fix the deuteron binding energy, or equivalently γ , and the asymptotic D/S ratio η by their experimental values. This fixes the short distance phases $\varphi_+(\gamma)$ and $\varphi_-(\gamma)$. Next, if we use an α or β (see below for a definition) zero energy scattering state we have in principle two short distance phases ($\varphi_{\alpha,+}(0), \varphi_{\alpha,-}(0)$) and ($\varphi_{\beta,+}(0), \varphi_{\beta,-}(0)$) which can be related to the (α_0, α_{02}) and (α_{02}, α_2) scattering lengths respectively. Using the orthogonality constraints to the deuteron bound state one can then eliminate α_{02} and α_2 and treat α_0 as a free parameter. Thus, in the triplet ${}^3S_1 - {}^3D_1$ channel we can treat γ , η and α_0 as independent parameters. Once these parameters have been fixed we can actually predict the corresponding phase shifts since any positive energy state must be orthogonal both to the deuteron bound state and the zero energy scattering states. This result is a direct consequence of the singular Van der Waals attractive behavior of the TPE potential at the origin. It is remarkable that this same set of independent parameters was also adopted in Ref. [54] within the realistic potential model treatment.

Conflicts between naive dimensional power counting and renormalization have been reported recently already at the LO (OPE) level [42] where it is shown that even the 3P_0 partial wave depends strongly on the cut-off in momentum space (a gaussian regulator is used) if according to the standard Weinberg counting no counterterm is added. The requirement of renormalizability makes the promotion of one counter term unavoidable for channels which present an attractive singularity. This promotion is the minimal possible one compatible with finiteness, because in a coupled channel problem one could think in general of three counterterms. From this viewpoint the choice of just one counterterm in triplet channels is a

bit mysterious. Our discussion in coordinate space agrees with these authors in the OPE potential, and actually allows to identify *a priori* the necessarily promotable counterterms as non trivial boundary conditions at the origin for singular attractive potentials. Moreover, we also see that the promotion of *only one counterterm* in the triplet channels with an attractive-repulsive singularity invoked in Ref. [42] is also maximal, since any additional counterterm would also produce divergent results (see Sect. VIII C). Thus, we see that although power counting determines the long distance potential, the short distance singular character of the potential does not allow to fix the counterterms arbitrarily.

To conclude this discussion, let us mention that the short distance phases, whenever they become relevant play the role of some dimensionless constants which depend exclusively on the form of the potential, but not on the potential parameters [41]. For the same reason they can be taken to be zeroth order in the power counting used to generate the chiral potential in Eq. (1), although they are subjected in general to higher order corrections. In this sense, the form of the short distance interaction is dictated by the potential only, and cannot be considered as independent information (See also Ref. [55] for a similar view on the three body problem in the absence of long distance potentials).

IV. THE SINGLET 1S_0 -CHANNEL

For the singlet 1S_0 channel one has to solve

$$-u''(r) + U_{1S_0}(r)u(r) = k^2 u(r) \quad (33)$$

At short distances the NNLO NN chiral potential behaves as [7, 11, 12]

$$\begin{aligned} U_{1S_0}(r) &\rightarrow \frac{3g^2}{128f^4\pi^2r^6}(-4 + 15g^2 + 24\bar{c}_3 - 8\bar{c}_4) \\ &= -\frac{R^4}{r^6} \end{aligned} \quad (34)$$

which is a Van der Waals type interaction with typical length scale $R = (-MC_6)^{1/4}$. Here, $\bar{c}_i = MC_i$. The value of the coefficient is negative for the four parameter sets of Table I, so the solution at short distances is of oscillatory type, Eq. (21) with $n = 6$, and

$$u(r) \rightarrow A \left(\frac{r}{R}\right)^{3/2} \sin \left[-\frac{1}{2} \left(\frac{R}{r}\right)^2 + \varphi \right] \quad (35)$$

where there is a undetermined energy independent phase, φ , and A is a normalization constant. Note that the corresponding Van der Waals radius is quite sensitive to the choice of chiral parameters.

considered. See the discussion at the end of Sect. V in Ref. [41]

TABLE I: Short distance Van der Waals coefficients for the NNLO chiral potential in singlet 1S_0 and the $^3S_1 - ^3D_1$ triplet channels.

Set	Source	$c_1(\text{GeV}^{-1})$	$c_3(\text{GeV}^{-1})$	$c_4(\text{GeV}^{-1})$	$MC_6(\text{fm}^4)$	$MC_{6,+}(\text{fm}^4)$	$MC_{6,-}(\text{fm}^4)$	θ (degrees)
Set I	(BM) πN [31]	-0.81 ± 0.15	-4.69 ± 1.34	3.40 ± 0.04	-8.74	-16.96	-5.63	140.8
Set II	(RTdS) NN [11]	-0.76	-5.08	4.70	-10.19	-21.45	-5.58	170.0
Set III	(EMa) NN [20]	-0.81	-3.40	3.40	-6.45	-14.68	-3.35	140.7
Set IV	(EMb) NN [28]	-0.81	-3.20	5.40	-7.28	-20.18	-1.86	182.6

A. Low energy parameters

For the zero energy state we use the asymptotic normalization at large distances

$$u_0(r) \rightarrow 1 - \frac{r}{\alpha_0}. \quad (36)$$

Then, the effective range is given by

$$r_0 = 2 \int_0^\infty dr \left[\left(1 - \frac{r}{\alpha_0} \right)^2 - u_0(r)^2 \right] \quad (37)$$

We can use the superposition principle for boundary conditions

$$u_0(r) = u_{0,c}(r) - \frac{1}{\alpha_0} u_{0,s}(r) \quad (38)$$

where $u_{0,c}(r) \rightarrow 1$ and $u_{0,s}(r) \rightarrow r$ correspond to cases where the scattering length is either infinity or zero respectively. Using this decomposition one gets

$$r_0 = A + \frac{B}{\alpha_0} + \frac{C}{\alpha_0^2}, \quad (39)$$

where

$$A = 2 \int_0^\infty dr (1 - u_{0,c}^2), \quad (40)$$

$$B = -4 \int_0^\infty dr (r - u_{0,c} u_{0,s}), \quad (41)$$

$$C = 2 \int_0^\infty dr (r^2 - u_{0,s}^2), \quad (42)$$

depend on the potential parameters only. The interesting thing is that all explicit dependence on the scattering length α_0 is displayed by Eq. (39). In a sense this is the non-perturbative universal form of a low energy theorem, which applies to *any* potential regular or singular at the origin which decays faster than a certain power of r at large distances (for an analytical example with the pure Van der Waals potential $U = -R^4/r^6$ see Sect. VII). Since the potential is known accurately at long distances we can visualize Eq. (39) as a long distance correlation between r_0 and α_0 . Naturally, if there is scale separation between the different contributions in the potential, Eq. (1), we expect the coefficients A,B and C to display a converging pattern. This is exactly what happens

(see Eq. (43) and Eq. (89) below) although not compatible with a naive and perturbative power counting (see Sect. IX).

In the 1S_0 the TPE potential becomes singular and attractive at short distances. Nevertheless, already at this point one can see the dramatic difference between attractive and repulsive singular potentials. In the attractive case the short distance phase allows to choose the scattering length *independently* on the potential, hence the coefficients A,B and C are uncorrelated with α_0 . For a singular repulsive potential, however, α_0 as well as A,B and C are determined by the potential. If one assumes A,B and C to be independent on α_0 in the repulsive case, this can only be possible due to an admixture of both the regular and irregular solutions, the latter will dominate at short distances and the effective range will diverge $r_0 \rightarrow -\infty$. This fact will become relevant in Sect. VIII C.

Obviously, for the chiral TPE potential, Eq. (1), the coefficients have to be evaluated by numerical means and they are finite. We expect that these coefficients scale with the relevant scale of the potential. If long distances dominate $A \sim 1/m$, $B \sim 1/m^2$ and $C \sim 1/m^3$ but then $r_0 \sim 1/m$. On the contrary if short distances dominate $A \sim R$, $B \sim R^2$ and $C \sim R^3$ and $r_0 \sim R$. The real situation is somewhat in between, but it is clear that A is far more sensitive to short distances than C. Actually, for a large scattering length, as it is the case in the 1S_0 channel, the coefficient A dominates. Note that unlike the standard approaches, where a short distance contribution to the effective range is allowed (in the form of a momentum dependent counterterm $C_2(k^2 + k'^2)$), we build r_0 *solely* from the potential and the scattering length α_0 . This is a direct consequence of the orthogonality relations, which preclude energy dependent boundary conditions for the local and energy independent chiral TPE potential.

In Fig. (3) we show the dependence of the effective range as a function of the short distance cut-off radius r_c , i.e. replacing the lower limit of integration in Eq. (37), for values between 2 and 0.1fm and taking the experimental value of the scattering length $\alpha_0 = -23.74\text{fm}$. As we see, the short distance behaviour is well under control and nicely convergent towards the experimental value. This dependence also illustrates that an error estimate based on varying the cut-off between certain range is only a measure on the size of finite cut-off effects, rather than a measure on the error. The linear behaviour observed

TABLE II: Threshold parameters in the singlet 1S_0 channel for the different sets of parameters c_1 , c_3 and c_4 given in Table I. We compare our renormalized results given by the cut-off independent universal formula (39) for r_0 and its extension for v_2 to finite cut-off NN calculations using their scattering length as an input. The difference is attributed to finite cut-off effects.

Set	Calculation	$\Lambda = 1/r_c$	$\alpha_0(\text{fm})$	$r_0(\text{fm})$	$v_2(\text{fm}^3)$
Set I	NNLO [10]	0.6-1 GeV	-23.72	2.68	-0.61
Set I	NNLO [19]	0.5 GeV	-23.75	2.70	-
Set I	This work	∞	Input	2.92	-0.30
Set II	NNLO [11]	1/1.4 fm	-	-	-
Set II	This work	∞	Input	2.97	-0.23
Set III	NNLO [23]	0.65 GeV	-23.4	2.67	-0.50
Set III	N ³ LO [29]	0.7 GeV	-23.6	2.66	-0.50
Set III	This work	∞	Input	2.83	-0.43
Set IV	N ³ LO [28]	0.5 GeV	-23.73	2.73	-
Set IV	This work	∞	Input	2.87	-0.38
Nijm II	[43, 44]	-	-23.73	2.67	-0.48
Reid 93	[43, 44]	-	-23.74	2.75	-0.49
Exp.	-	-	-23.74(2)	2.77(5)	-

at small r_c is a consequence of the dominance of the first term in Eq. (37) as compared to the second term where the wave function contribution vanishes as $\sim r_c^4$. Let us remind that in the conventional treatments a counterterm C_2 is added to provide a short distance contribution to the effective range parameter and the result is fitted to experiment so that r_0 becomes an input of the calculation. Obviously one expects C_2 to depend on the regularization scale. As shown in Fig. (3) the size of the counterterm C_2 at $r_c \rightarrow 0$ must be numerically small, since the TPE potential provides the bulk of the contribution. This agrees with the orthogonality constraint which requires $C_2 \rightarrow 0$ when $r_c \rightarrow 0$.

Numerically we find the following renormalized relations in the singlet channel for the OPE and NNLO TPE,

$$\begin{aligned}
 r_0 &= 1.308062 - \frac{4.547741}{\alpha_0} + \frac{5.192606}{\alpha_0^2} & (\text{OPE}), \\
 r_0 &= 2.670963 - \frac{5.755234}{\alpha_0} + \frac{6.031119}{\alpha_0^2} & (\text{Set I}), \\
 r_0 &= 2.715075 - \frac{5.847358}{\alpha_0} + \frac{6.093430}{\alpha_0^2} & (\text{Set II}), \\
 r_0 &= 2.586862 - \frac{5.584383}{\alpha_0} + \frac{5.916900}{\alpha_0^2} & (\text{Set III}), \\
 r_0 &= 2.616830 - \frac{5.640921}{\alpha_0} + \frac{5.952694}{\alpha_0^2} & (\text{Set IV})
 \end{aligned} \tag{43}$$

As we see, the coefficient dependent of α_0^{-2} is not very sensitive to the choice of the coefficients c_1 , c_3 , c_4 and the OPE potential already provides the bulk of the contribution. On the other hand, the coefficient independent on

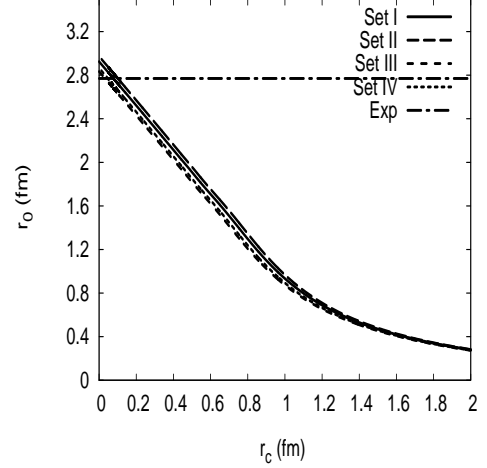


FIG. 3: Effective range for the TPE potential in the singlet 1S_0 channel when the lower limit of integration in Eq. (37) is taken to be a short distance cut-off and the experimental value of the scattering length $\alpha_0 = -23.74\text{fm}$ is taken. We show Sets I,II,III and IV (see main text).

α_0 changes dramatically when going from OPE to TPE, suggesting that the effect is clearly non-perturbative. A direct inspection of the integrands for the A,B and C coefficients shows that A picks its main contribution from the short distance region around 1 fm, whereas for B and C the most important contribution is located around 3 fm. One expects that different choices of coefficients c_3 and c_4 influence mostly the A coefficient. We confirm this expectation analytically by only keeping the Van der Waals contribution to the full potential in Sect. VII. We emphasize, again, that A , B and C are intrinsic information of the potential; these values for the effective range stem solely from the NNLO chiral potential and the scattering length α_0 , without any additional short distance contribution. The closeness of these numbers to the experimental value suggests that there is perhaps no need to make the boundary condition energy dependent if the cut-off is indeed removed, and that the missing 0.1fm contribution can be clearly attributed to N³LO contributions in the potential.

The results are summarized in Table II. For pn we have the experimental values $\alpha_0 = -23.74(2)$ and $r_0 = 2.77(5)$. The previous formula, Eq. (39) yields $r_0 = 2.92, 2.97, 2.83, 2.87$ for Sets I,II, III and IV respectively, which show a systematic discrepancy with the published values in several works (see References at the Table II) and also a systematic trend to discrepancy with respect to the experimental value. Our renormalized values are always larger than the finite cut-off results. This seems natural since finite cut-off corrections diminish the integration region. Note also that the size of the discrepancy is *larger* than the experimental uncertainties and hence is statistically significant.

The value of the effective range was not given in the

coordinate space calculation of Ref. [11] but the quality of the fit suggests that they get a value very close to the experimental one, $r_0 = 2.75$. The contribution to the effective range from the origin to 0.1fm is about 0.2. In Ref. [11] the cut-off is in coordinate space and an energy dependent boundary condition is considered. This means in practice cutting-off the lower integration in Eq. (37) at $a = 1.4\text{fm}$ and adding a short distance contribution r_S as to reproduce the experimental value. This introduces a new potential independent parameter. As we have argued, in the limit $a \rightarrow 0$, the short distance contribution of the effective range should go to zero, as implied by the orthogonality constraints. For finite a , the orthogonality constraint does not imply a vanishing short distance contribution to the effective range.

For Set IV one could reach the upper experimental value by flipping the sign of c_1 and keeping c_3 and c_4 unchanged. For $c_1 = 2.43\text{GeV}^{-1}$ one gets $r_0 = 2.78\text{fm}$. The full experimental range would be covered by letting $0.81\text{GeV}^{-1} < c_1 < 4.90\text{GeV}^{-1}$. This is in total contradiction to the expectations of πN scattering studies [31]. The insensitivity of our results with respect to the c_1 coefficient has to do with the fact that c_1 only enters in the potential at short distances at order $1/r^4$ which is sub-leading as compared to the leading Van der Waals singularity. This is another confirmation on the short distance dominance in the effective range parameter r_0 .

Thus, according to our analysis, for the accepted values of chiral constants of Sets I,II, III and IV used in previous works, the difference in the value of the 1S_0 effective range could only be attributed to three pion exchange, relativistic effects and electromagnetic corrections. Another possibility, of course, is to refit the chiral constants to our renormalized, cut-off free results. This will be discussed in Sect. VI.

B. Phase Shift

For a finite energy scattering state we solve for the chiral TPE potential with the normalization

$$u_k(r) \rightarrow \frac{\sin(kr + \delta_0)}{\sin \delta_0} \quad (44)$$

Again, if we use the superposition principle

$$u_k(r) = u_{k,c}(r) + k \cot \delta_0 u_{k,s}(r) \quad (45)$$

with $u_{k,c} \rightarrow \cos(kr)$ and $u_{k,s} \rightarrow \sin(kr)/k$ and impose the orthogonality constraint with the zero energy state to get

$$\frac{u'_{k,c}(a) + k \cot \delta_0 u'_{k,s}(a)}{u_{k,c}(a) + k \cot \delta_0 u_{k,s}(a)} = \frac{-\alpha_0 u'_{0,c}(a) + u'_{0,s}(a)}{-\alpha_0 u_{0,c}(a) + u_{0,s}(a)} \quad (46)$$

Note that the dependence of the phase-shift on the scattering length is *explicit*; $\cot \delta_0$ is a bilinear rational mapping of α_0 . Taking the limit $a \rightarrow 0$ we get

$$k \cot \delta_0 = \frac{\alpha_0 \mathcal{A}(k) - \mathcal{B}(k)}{\alpha_0 \mathcal{C}(k) - \mathcal{D}(k)} \quad (47)$$

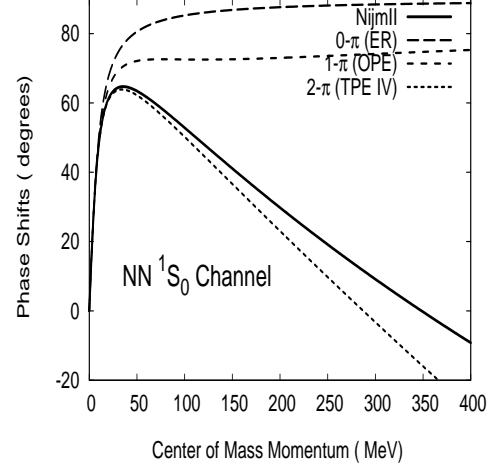


FIG. 4: Renormalized phase shifts for the OPE and TPE potentials as a function of the CM np momentum k in the singlet 1S_0 channel compared to the Nijmegen results [43] for different parameter sets. The regular scattering wave functions for finite k are orthogonal to the zero energy wave functions. For the TPE potential we have taken the chiral couplings of set IV.

whereas the functions \mathcal{A} , \mathcal{B} , \mathcal{C} and \mathcal{D} are even functions of k which depend only on the potential and are given by

$$\begin{aligned} \mathcal{A}(k) &= \lim_{a \rightarrow 0} (u_{0,c}(a)u'_{k,c}(a) - u'_{0,c}(a)u_{k,c}(a)) \\ \mathcal{B}(k) &= \lim_{a \rightarrow 0} (u_{k,c}(a)u'_{0,s}(a) - u_{0,s}(a)u'_{k,c}(a)) \\ \mathcal{C}(k) &= \lim_{a \rightarrow 0} (u'_{0,c}(a)u_{k,s}(a) - u_{0,c}(a)u'_{k,s}(a)) \\ \mathcal{D}(k) &= \lim_{a \rightarrow 0} (u_{0,s}(a)u'_{k,s}(a) - u'_{0,s}(a)u_{k,s}(a)) \end{aligned} \quad (48)$$

The obvious conditions $\mathcal{A}(0) = \mathcal{D}(0) = 0$ and $\mathcal{B}(0) = \mathcal{C}(0) = 1$ are satisfied. Expanding the expression for small k one gets the well known effective range expansion

$$k \cot \delta = -\frac{1}{\alpha_0} + \frac{1}{2}r_0 k^2 + v_2 k^4 + \dots \quad (49)$$

where v_k is a polynomial in $1/\alpha_0$ of degree $k+1$.

The renormalized phase shift is presented in Fig. 4 for Set IV. As we see the trend in the effective range r_0 and the v_2 parameter is reflected in the behavior of the phase shift ¹⁰

¹⁰ Let us mention that the momentum space calculation of Ref. [42] does not reproduce the physical and well measured scattering length. We have checked by an explicit solution of the Lipmann-Schwinger equation that the problem is more related to an insufficient number of Gauss points rather than to the value of the cut-off.

V. THE TRIPLET ${}^3S_1 - {}^3D_1$ CHANNEL

The coupled channel ${}^3S_1 - {}^3D_1$ set of equations read

$$\begin{aligned} -u''(r) + U_{3S_1}(r)u(r) + U_{E_1}(r)w(r) &= k^2 u(r), \\ -w''(r) + U_{E_1}(r)u(r) + \left[U_{3D_1}(r) + \frac{6}{r^2}\right]w(r) &= k^2 w(r), \end{aligned} \quad (50)$$

At short distances the NN chiral NNLO potential behaves as [7, 11, 12]

$$\begin{aligned} U_{3S_1}(r) &\rightarrow \frac{MC_{6,3S_1}}{r^6} \\ U_{E_1}(r) &\rightarrow \frac{MC_{6,E_1}}{r^6} \\ U_{3D_1}(r) &\rightarrow \frac{MC_{6,3D_1}}{r^6} \end{aligned} \quad (51)$$

which is a coupled channels Van der Waals type interaction where the coefficients are given by

$$\begin{aligned} MC_{3S_1} &= \frac{3g^2}{128f^4\pi^2}(4 - 3g^2 + 24\bar{c}_3 - 8\bar{c}_4) \\ MC_{E_1} &= -\frac{3\sqrt{2}g^2}{128f^4\pi^2}(-4 + 3g^2 - 16\bar{c}_4) \\ MC_{3D_1} &= \frac{9g^2}{32f^4\pi^2}(-1 + 2g^2 + 2\bar{c}_3 - 2\bar{c}_4) \end{aligned} \quad (52)$$

If we diagonalize the corresponding matrix we get

$$\begin{aligned} \begin{pmatrix} C_{6,3S_1} & C_{6,E_1} \\ C_{6,E_1} & C_{6,3D_1} \end{pmatrix} &= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} C_{6,+} & 0 \\ 0 & C_{6,-} \end{pmatrix} \\ &\times \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \end{aligned} \quad (53)$$

where $C_{6,\pm}$ are the corresponding eigenvalues and θ the mixing angle. They are listed in Table I for different parameters choices of the chiral couplings c_1 , c_3 and c_4 . We see that in all cases both eigenpotentials are attractive at short distances and hence the short distance behavior of the wave functions is of oscillatory type with $n = 6$. Defining the Van der Waals scales

$$R_{\pm} = (-MC_{6,\pm})^{1/4} \quad (54)$$

the short distance solutions read

$$\begin{pmatrix} u \\ w \end{pmatrix} \rightarrow \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \left(\frac{r}{R_+}\right)^{\frac{3}{2}} \sin\left[\frac{1}{2}\left(\frac{R_+}{r}\right)^2 + \varphi_+\right] \\ \left(\frac{r}{R_-}\right)^{\frac{3}{2}} \sin\left[\frac{1}{2}\left(\frac{R_-}{r}\right)^2 + \varphi_-\right] \end{pmatrix}$$

Thus, we have two arbitrary short distance phases φ_{\pm} for a given fixed energy which cannot be deduced from the

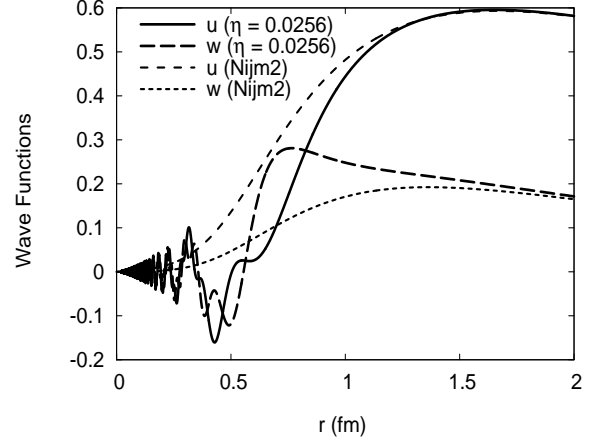


FIG. 5: The TPE deuteron wave functions, u and w , as a function of the distance (in fm) compared to the Nijmegen II wave functions [43, 44]. The asymptotic normalization $u \rightarrow e^{-\gamma r}$ has been adopted and the asymptotic D/S ratio is taken $\eta = 0.0256(4)$. We use the set IV of chiral couplings.

potential and hence have to be treated as independent parameters. We will fix them to some physical observables by integrating Eqs. (50) from infinity down to the origin.

A. The deuteron

In the deuteron $k^2 = -\gamma^2$ and we solve Eq. (50) together with the asymptotic condition at infinity

$$\begin{aligned} u(r) &\rightarrow A_S e^{-\gamma r}, \\ w(r) &\rightarrow A_D e^{-\gamma r} \left(1 + \frac{3}{\gamma r} + \frac{3}{(\gamma r)^2}\right), \end{aligned} \quad (55)$$

where $\gamma = \sqrt{MB}$ is the deuteron wave number, A_S is the normalization factor and the asymptotic D/S ratio parameter is defined by $\eta = A_D/A_S$. In what follows we use γ and η as *input* parameters thus fixing the short distance phases φ_{\pm} automatically.

In this paper we compute the matter radius, which reads,

$$r_m^2 = \frac{\langle r^2 \rangle}{4} = \frac{1}{4} \int_0^\infty r^2 (u(r)^2 + w(r)^2) dr, \quad (56)$$

the quadrupole moment (without meson exchange currents)

$$Q_d = \frac{1}{20} \int_0^\infty r^2 w(r) (2\sqrt{2}u(r) - w(r)) dr, \quad (57)$$

the deuteron inverse radius

$$\langle r^{-1} \rangle = \int_0^\infty dr \frac{u(r)^2 + w(r)^2}{r}, \quad (58)$$

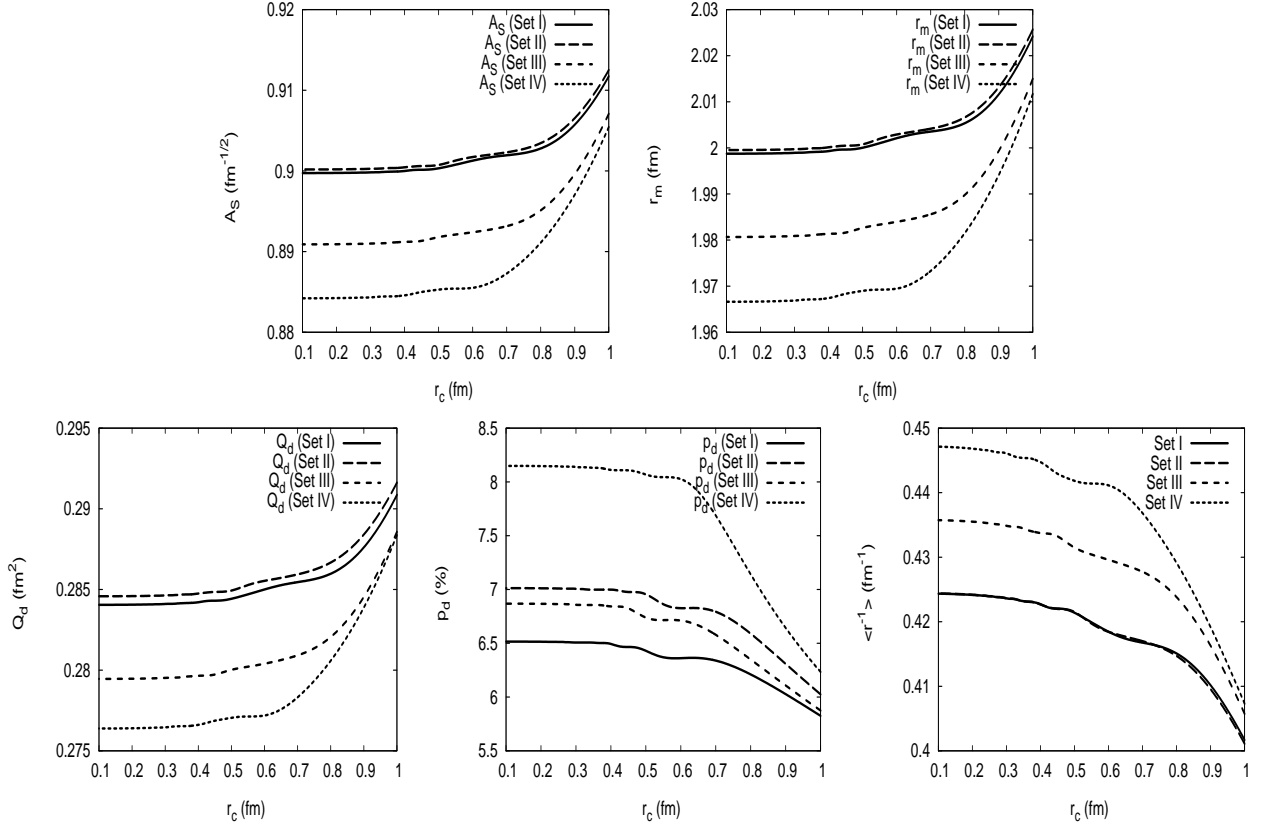


FIG. 6: The dependence of the S-wave normalization A_S (in $\text{fm}^{-1/2}$, upper left panel), the matter radius r_m (in fm, upper right panel), the quadrupole moment Q_d (in fm^2 , lower left panel), the D -state probability (lower mid panel) and the inverse radius $\langle r^{-1} \rangle$ (in fm^{-1} lower right panel) for the TPE potential on the short distance cut-off, r_c (in fm) for Sets I,II,III and IV of low energy chiral constants.

which appears in low energy pion-deuteron scattering, and the D -state probability

$$P_D = \int_0^\infty w(r)^2 dr. \quad (59)$$

Following Ref. [41] we use the superposition principle of boundary conditions and write

$$\begin{aligned} u(r) &= u_S(r) + \eta u_D(r), \\ w(r) &= w_S(r) + \eta w_D(r), \end{aligned} \quad (60)$$

where (u_S, w_S) and (u_D, w_D) correspond to the boundary conditions at infinity, Eq. (55), with $A_S = 1$ and $A_D = 0$ and with $A_S = 0$ and $A_D = 1$ respectively. Obviously, u_S, u_D, w_S and w_D depend on the potential and the deuteron binding energy only, so that the dependence on the asymptotic D/S ratio η can be determined analytically. The value is taken as a free parameter. The resulting deuteron wave functions for Set IV are displayed in Fig. 5 and compared to the Nijmegen II results [43, 44]. One clearly sees the incommensurable ever increasing oscillations already below $r = 0.6\text{fm}$.

The short distance cut-off dependence of these deuteron properties using the experimental values for the

deuteron binding energies and the asymptotic D/S ratio, $\eta = 0.0256$, can be looked up in Fig. 6. As one sees the cut-off dependence is well under control, so the infinite cut-off limit can be extracted without difficulty.

Using the superposition principle of boundary conditions, Eq. (60) the asymptotic S-wave ratio depends quadratically on η as follows

$$\begin{aligned} \frac{1}{A_S^2} &= \int_0^\infty dr (u_S^2 + w_S^2) + 2\eta \int_0^\infty dr (u_S u_D + w_S w_D) \\ &+ \eta^2 \int_0^\infty dr (u_D^2 + w_D^2). \end{aligned} \quad (61)$$

The coefficients of this second order polynomial depends on the potential and the deuteron binding energy. Similar relations hold for other observables. Evaluating the integrals numerically we get the following analytic correlations

Set I

$$\begin{aligned}
1/A_S^2 &= 3.78888 - 214.675 \eta + 4489.43 \eta^2 \\
r_m^2/A_S^2 &= 5.47297 - 54.1956 \eta + 1295.89 \eta^2 \\
Q_d/A_S^2 &= -0.342883 + 36.6449 \eta - 372.841 \eta^2 \\
P_D/A_S^2 &= 2.10904 - 184.824 \eta + 4124.37 \eta^2 \\
\langle r^{-1} \rangle/A_S^2 &= 3.58173 - 252.20 \eta + 5186.53 \eta^2 \quad (62)
\end{aligned}$$

Set II

$$\begin{aligned}
1/A_S^2 &= 3.01271 - 155.591 \eta + 3363.94 \eta^2 \\
r_m^2/A_S^2 &= 5.34737 - 44.8896 \eta + 1122.59 \eta^2 \\
Q_d/A_S^2 &= -0.296852 + 33.2406 \eta - 309.624 \eta^2 \\
P_D/A_S^2 &= 1.44293 - 132.314 \eta + 3098.89 \eta^2 \\
\langle r^{-1} \rangle/A_S^2 &= 2.52815 - 171.36 \eta + 3635.18 \eta^2 \quad (63)
\end{aligned}$$

Set III

$$\begin{aligned}
1/A_S^2 &= 4.65049 - 283.545 \eta + 5902.53 \eta^2 \\
r_m^2/A_S^2 &= 5.58929 - 63.1854 \eta + 1481.50 \eta^2 \\
Q_d/A_S^2 &= -0.377779 + 39.2691 \eta - 420.250 \eta^2 \\
P_D/A_S^2 &= 2.77521 - 241.491 \eta + 5330.67 \eta^2 \\
\langle r^{-1} \rangle/A_S^2 &= 4.87639 - 356.21 \eta + 7311.77 \eta^2 \quad (64)
\end{aligned}$$

Set IV

$$\begin{aligned}
1/A_S^2 &= 3.40962 - 190.713 \eta + 4198.86 \eta^2 \\
r_m^2/A_S^2 &= 5.40066 - 49.2912 \eta + 1232.81 \eta^2 \\
Q_d/A_S^2 &= -0.306469 + 33.9354 \eta - 318.598 \eta^2 \\
P_D/A_S^2 &= 1.66525 - 155.233 \eta + 3681.89 \eta^2 \\
\langle r^{-1} \rangle/A_S^2 &= 3.14658 - 226.20 \eta + 4907.35 \eta^2 \quad (65)
\end{aligned}$$

The numerical coefficients in these expressions depend on the deuteron binding energy and the TPE potential parameters, g , m , f , c_1 , c_3 and c_4 . The results for the deuteron properties are given in Table III. The uncertainties are due to changing the input γ and η within their experimental uncertainties. We have checked that the short distance cut-offs $a \sim 0.1 - 0.2 \text{ fm}$ generates much smaller uncertainties. The explicit dependence on η is displayed in Fig. 7. Again, we find a discrepancy in the case of Set I with the values quoted in the finite cut-off calculation. Remarkably, our renormalized results in coordinate space agree most with the momentum space calculation of Ref. [19] corresponding to Set IV. It is noticeable that this can be done without explicit knowledge of the counterterms used in that work in momentum space. This is precisely one of the points of renormalization; results can be reproduced by just providing physical input data, and no particular reference to the method of solution. Let us remind that the c_1 , c_3 and c_4 were fixed from the perturbative study of NN peripheral waves where the cut-off sensitivity is rather small. Nevertheless, some significant discrepancies do also occur.

For the parameter Set IV [28] obtained by a N³LO fit to NN scattering data, our NNLO calculation almost

reproduces exactly the numbers provided in that work. Furthermore, they turn out to be compatible with the experimental numbers at the 1σ level within the uncertainty induced by the asymptotic D/S ratio¹¹

One immediate lesson we learn from inspection of Table III is that, regardless of the parameter set, only the experimental uncertainty in the asymptotic D/S ratio for the deuteron generates theoretical uncertainties about an order of magnitude larger than the experimental ones. On top of this, one has also to take into account other uncertainties, such as the one in $g_{\pi NN}$ and, of course, those induced by c_1 , c_3 and c_4 , which generally will generate larger uncertainties if all these parameters are regarded as independent (see Sect. VI below). In addition, there are systematic errors related to the accuracy of the expansion in the potential, Eq. (1). In common with non-perturbative finite cut-off calculations [10, 11, 21, 22, 23, 28] they are difficult to estimate *a priori* given the non-perturbative nature of our calculation, but are bound to increase the error (see, however, our discussion in Sect. IX below on non-integer power counting). Given the insensitivity of our results with the short distance cut-off, the procedure used in Ref. [22, 23] of varying the cut-off becomes unsuitable in our case.

For the deuteron channel one may conclude that the predictive power of the chiral expansion has reached a limit at NNLO. So, at present, we do not expect to make theoretical predictions in the deuteron to be more accurate than experiment. The inclusion of N³LO and higher orders may provide better central values but is unlikely to improve the situation regarding error estimates since new unknown coefficients in the potential appear and the induced uncertainties will generally increase.

On the other hand, the slope for A_S and r_m , Fig. 7, suggests that it would be better to take the asymptotic S-wave normalization or the matter radius as input, since generated errors may be comparable or even smaller. For instance, if the matter radius r_m is taken as input we get instead $\eta = 0.0253(4)$ a compatible value with similar errors. However, if we take $A_S = 0.8846(9)$ as input for Set IV we get $\eta = 0.0255(1)$ a compatible value with the experimental one but with much smaller errors. The reduction of errors is also confirmed in Sets I, II and III, although the central values are a bit off. This result opens up the possibility of making a benchmark determination of the asymptotic D/S deuteron ratio from the chiral effective theory. Obviously, to do so, the chiral constants should be known with rather high accuracy, an illusory

¹¹ One may object that one should not use N³LO parameters to do a NNLO calculation, since they are obtained by fitting the same database. However, if there are finite cut-off effects the situation is not as clear. Finite cut-off effects are minimized in a N³LO calculation as shown in Ref. [29] where the induced uncertainties are drastically reduced when going from NNLO to N³LO. Note that in our calculation there are no sizeable cut-off induced uncertainties already at NNLO.

expectation at the present moment. In this regard it would perhaps be profitable to pin down the errors for the chiral constants from peripheral waves. This point will be analyzed elsewhere [56].

Both the loss of predictive power and the very rare possibility of making model independent theoretical predictions for purely hadronic processes using Chiral Perturbation Theory more accurate than experiment we seem to observe in low energy NN scattering is not new and has already been documented for low energy $\pi\pi$ scattering [49, 50, 51] and provides a further motivation to use chiral effective approaches.

B. Low energy parameters

The zero energy wave functions are taken asymptotically as ¹²

$$\begin{aligned} u_{0,\alpha}(r) &\rightarrow 1 - \frac{r}{\alpha_0}, \\ w_{0,\alpha}(r) &\rightarrow \frac{3\alpha_{02}}{\alpha_0 r^2}, \\ u_{0,\beta}(r) &\rightarrow \frac{r}{\alpha_0}, \\ w_{0,\beta}(r) &\rightarrow \left(\frac{\alpha_2}{\alpha_{02}} + \frac{\alpha_{02}}{\alpha_0} \right) \frac{3}{r^2} - \frac{r^3}{15\alpha_{02}}. \end{aligned} \quad (66)$$

Using these zero energy solutions one can determine the effective range. The 3S_1 effective range parameter is given by

$$r_0 = 2 \int_0^\infty \left[\left(1 - \frac{r}{\alpha_0} \right)^2 - u_\alpha(r)^2 - w_\alpha(r)^2 \right] dr. \quad (67)$$

In the zero energy case, the vanishing of the diverging exponentials at the origin imposes a condition on the α and β states which generate a correlation between α_0 , α_{02} and α_2 . Using the superposition principle of boundary conditions we may write the solutions in such a way

¹² We correct an error in Eq.(45) of our previous work [41] where α_2 appears. The corrected numerical value is $\alpha_2 = 6.693\text{fm}^5$.

that

$$\begin{aligned} u_{0,\alpha}(r) &= u_1(r) - \frac{1}{\alpha_0} u_2(r) + \frac{3\alpha_{02}}{\alpha_0} u_3(r) \\ w_{0,\alpha}(r) &= w_1(r) - \frac{1}{\alpha_0} w_2(r) + \frac{3\alpha_{02}}{\alpha_0} w_3(r) \\ u_{0,\beta}(r) &= \frac{1}{\alpha_0} u_2(r) + \left(\frac{3\alpha_2}{\alpha_{02}} + \frac{3\alpha_{02}}{\alpha_0} \right) u_3(r) - \frac{1}{15\alpha_{02}} u_4(r) \\ w_{0,\beta}(r) &= \frac{1}{\alpha_0} w_2(r) + \left(\frac{3\alpha_2}{\alpha_{02}} + \frac{3\alpha_{02}}{\alpha_0} \right) w_3(r) - \frac{1}{15\alpha_{02}} w_4(r) \end{aligned} \quad (68)$$

where the functions $u_{1,2,3,4}$ and $w_{1,2,3,4}$ are independent on α_0 , α_{02} and α_2 and fulfill suitable boundary conditions. The orthogonality constraints for the α and β states read in this case

$$\begin{aligned} u_\gamma u'_{0,\alpha} - u'_\gamma u_{0,\alpha} + w_\gamma w'_{0,\alpha} - w'_\gamma w_{0,\alpha} \Big|_{r=r_c} &= 0 \\ u_\gamma u'_{0,\beta} - u'_\gamma u_{0,\beta} + w_\gamma w'_{0,\beta} - w'_\gamma w_{0,\beta} \Big|_{r=r_c} &= 0 \end{aligned} \quad (69)$$

yielding two relations between γ , α_{02} , α_2 , η and α_0 , meaning that two of them are not independent. Using the superposition principle decomposition of the bound state, Eq. (60), and for the zero energy states, Eq. (68), we make the orthogonality relation explicit in α_0 , α_{02} , α_2 and η . If we would use α_0 , α_{02} , α_2 as input parameters the orthogonality constraint is actually a non-linear eigenvalue problem for γ and η . The values of α_{02} and α_2 are not so well known although they have been determined in potential models in our previous work [57]. In contrast, γ , η and α_0 are well determined experimentally. Thus, in the deuteron scattering channel we will use γ , η and α_0 as independent input parameters and α_{02} , α_2 as predictions. This same set of independent parameters was also adopted in Ref. [54] within the high quality potential model treatment, although the role of the short distance Van der Waals singularity was not recognized. Fixing the experimental value of γ we get the following relations for different parameter choices of c_1 , c_3 and c_4 ,

Set I

$$\begin{aligned} \alpha_{02} &= \frac{2.01763 - 0.456461 \alpha_0 - 44.8947 \eta + 11.9351 \alpha_0 \eta}{-0.314426 + 13.1555 \eta} \\ \alpha_2 &= \frac{-0.023522 + 1.04677 \eta - 11.6459 \eta^2 + \alpha_0 (0.008423 - 0.537856 \eta + 9.39376 \eta^2)}{\alpha_0 (-0.023901 + \eta)^2} + \frac{\alpha_{02}^2}{\alpha_0} \end{aligned} \quad (70)$$

Set II

$$\begin{aligned} \alpha_{02} &= \frac{1.71745 - 0.373228 \alpha_0 - 33.4616 \eta + 8.76639 \alpha_0 \eta}{-0.228075 + 9.865911 \eta} \\ \alpha_2 &= \frac{-0.030303 + 1.18083 \eta - 11.5032 \eta^2 + \alpha_0 (0.009559 - 0.566611 \eta + 9.71850 \eta^2)}{\alpha_0 (-0.023118 + \eta)^2} + \frac{\alpha_{02}^2}{\alpha_0} \end{aligned} \quad (71)$$

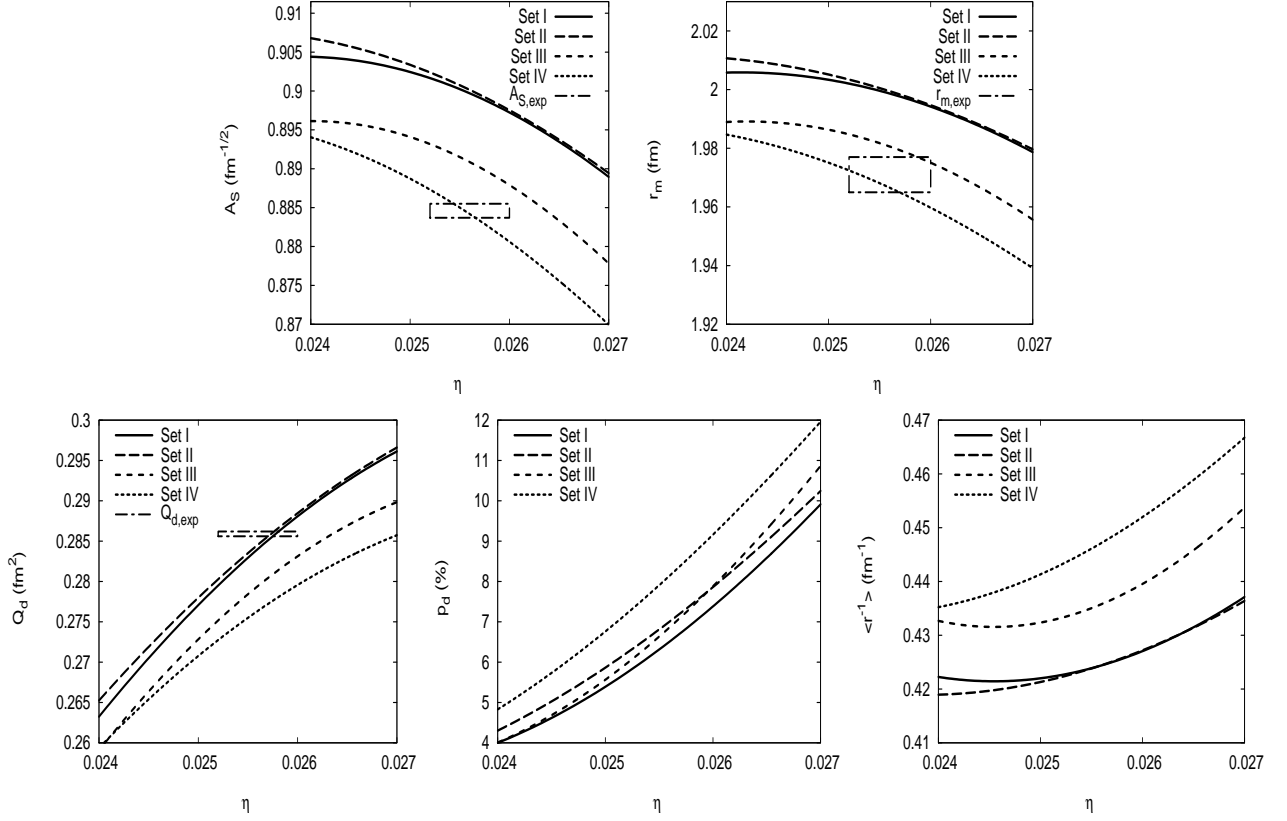


FIG. 7: The dependence of the S-wave normalization A_S (in $\text{fm}^{-1/2}$, upper left panel), the matter radius r_m (in fm, upper right panel), the quadrupole moment Q_d (in fm^2 , lower left panel) (lower mid panel) and the inverse radius $\langle r^{-1} \rangle$ (in fm^{-1} , lower right panel) for the TPE potential as a function of the asymptotic D/S ratio η . The boxes represent the experimental values. The predicted Quadrupole moments as well as the matter radius should be corrected for MEC's (accounting for adding 0.01fm^2 and 0.003fm respectively) on top of the potential result. We display the four sets of chiral coupling constants.

Set III

$$\alpha_{02} = \frac{2.36659 - 0.550871 \alpha_0 - 59.1666 \eta + 15.7488 \alpha_0 \eta}{-0.414962 + 17.2806 \eta}$$

$$\alpha_2 = \frac{-0.018755 + 0.937802 \eta - 11.7229 \eta^2 + \alpha_0 (0.007437 - 0.505434 \eta + 9.09925 \eta^2)}{\alpha_0 (-0.024013 + \eta)^2} + \frac{\alpha_{02}^2}{\alpha_0} \quad (72)$$

Set IV

$$\alpha_{02} = \frac{1.89526 - 0.418953 \alpha_0 - 41.8369 \eta + 10.8526 \alpha_0 \eta}{-0.279236 + 12.2978 \eta}$$

$$\alpha_2 = \frac{-0.023751 + 1.04857 \eta - 11.5733 \eta^2 + \alpha_0 (0.008050 - 0.518604 \eta + 9.31798 \eta^2)}{\alpha_0 (-0.023118 + \eta)^2} + \frac{\alpha_{02}^2}{\alpha_0} \quad (73)$$

The numerical coefficients appearing in these equations depend on the deuteron wave number γ and the TPE parameters, g, f, m and c_1, c_3 and c_4 . The dependence on η for fixed values of α_0 within its experimental uncertainty is depicted in Fig. 8. We see that for fixed chiral couplings c_1, c_3 and c_4 , the η uncertainty dominates the

errors. Numerical values can be seen at Table III. Note the large discrepancy in the effective range r_0 for Sets I and II with the experimental number. Finite cut-off effects are observed in Set III although the η induced uncertainty would make the value compatible with that estimate. Good agreement is observed again for Set IV,

TABLE III: Deuteron properties and low energy parameters in the $^3S_1 - ^3D_1$ channel for the pionless theory (Short), the OPE and the TPE potential. We use the non-relativistic relation $\gamma = \sqrt{2\mu_{np}B}$ with $B = 2.224575(9)$. For OPE* we take $g_A = 1.26$ as input instead of $g_{\pi NN} = 13.083$ as input. The errors quoted in the TPE reflect the uncertainty in the non-potential parameters γ , η and α_0 only. Differences from this work are attributed to finite cut-off effects. Experimental values can be traced from [54].

Set	Ref.	γ	η	A_S	r_m	Q_d	P_D	$\langle r^{-1} \rangle$	α_0	α_{02}	α_2	r_0
Short	-	Input	0	0.6806	1.5265	0	0	∞	4.3177	0	0	0
OPE	[41]	Input	0.02633	0.8681(1)	1.9351(5)	0.2762(1)	7.31(1)%	0.476(3)	5.335(1)	1.673(1)	6.693(1)	1.638(1)
OPE*	[41]	Input	0.02555	0.8625(2)	1.9234(5)	0.2667(1)	7.14(1)%	0.471(3)	5.308(1)	1.612(1)	6.325(1)	1.602(2)
Set I	N ² LO [10]		0.0245	0.884	1.967	0.262	6.11 %	-	5.420	-	-	1.753
Set I	N ² LO [19]		0.0256	0.8846	1.9756	0.281	4.17 %	-	5.417	-	-	1.753
Set I	This work	Input	Input	0.900(2)	1.999(4)	0.284(4)	6(1)%	0.424(3)	Input	2.3(2)	3(3)	1.4(4)
Set II	[11]	-	-	-	-	-	-	-	-	-	-	-
Set II	This work	Input	Input	0.900(2)	1.999(4)	0.285(4)	7(1)%	0.424(4)	Input	2.22(15)	4(2)	1.46(19)
Set III	N ² LO [23]	-	0.0256	0.873	1.972	0.272	5 %	-	5.427	-	-	1.731
Set III	N ³ LO [29]	-	0.0254	0.882	1.979	0.266	3 %	-	5.417	-	-	1.745
Set III	This work	Input	Input	0.891(3)	1.981(5)	0.279(4)	7(1)%	0.436(3)	Input	1.88(10)	5.7(16)	1.67(8)
Set IV	N ³ LO [28]	-	0.0256	0.8843	1.968	0.275	4.51 %	-	5.417	-	-	1.752
Set IV	This work	Input	Input	0.884(4)	1.967(6)	0.276(3)	8(1)%	0.447(5)	Input	1.67(4)	6.6(4)	1.76(3)
NijmII	[43, 44]	0.2316	0.02521	0.8845(8)	1.9675	0.2707	5.635%	0.4502	5.418	1.647	6.505	1.753
Reid93	[43, 44]	0.2316	0.02514	0.8845(8)	1.9686	0.2703	5.699%	0.4515	5.422	1.645	6.453	1.755
Exp.	-	0.2316	0.0256(4)	0.8846(9)	1.971(6)	0.2859(3)	-	-	5.419(7)	-	-	1.753(8)

particularly in the E_1 and 3D_1 scattering lengths and the effective range r_0 , but only the latter provides a clear TPE improvement as compared to OPE. The quantities α_{02} and α_2 are compatible with typical expectations [57] from the high quality potential models.

C. Phase Shifts

For the α and β positive energy scattering states we choose the asymptotic normalization

$$\begin{aligned}
u_{k,\alpha}(r) &\rightarrow \frac{\cos \epsilon}{\sin \delta_1} \left(\hat{j}_0(kr) \cos \delta_1 - \hat{y}_0(kr) \sin \delta_1 \right), \\
w_{k,\alpha}(r) &\rightarrow \frac{\sin \epsilon}{\sin \delta_1} \left(\hat{j}_2(kr) - \hat{y}_2(kr) \sin \delta_1 \right),
\end{aligned} \tag{74}$$

$$\begin{aligned}
u_{k,\beta}(r) &\rightarrow -\frac{1}{\sin \delta_1} \left(\hat{j}_0(kr) \cos \delta_2 - \hat{y}_0(kr) \sin \delta_2 \right), \\
w_{k,\beta}(r) &\rightarrow \frac{\tan \epsilon}{\sin \delta_1} \left(\hat{j}_2(kr) \cos \delta_2 - \hat{y}_2(kr) \sin \delta_2 \right),
\end{aligned} \tag{75}$$

where $\hat{j}_l(x) = x j_l(x)$ and $\hat{y}_l(x) = x y_l(x)$ are the reduced spherical Bessel functions and δ_1 and δ_2 are the eigen-phases in the 3S_1 and 3D_1 channels, and ϵ is the mixing angle E_1 . The use of the superposition principle for boundary conditions as well as the orthogonality con-

straints,

$$\begin{aligned}
u_\gamma u'_{k,\alpha} - u'_\gamma u_{k,\alpha} + w_\gamma w'_{k,\alpha} - w'_\gamma w_{k,\alpha} \Big|_{r=r_c} &= 0 \\
u_\gamma u'_{k,\beta} - u'_\gamma u_{k,\beta} + w_\gamma w'_{k,\beta} - w'_\gamma w_{k,\beta} \Big|_{r=r_c} &= 0
\end{aligned} \tag{76}$$

analogous to Eq. (69), to the deuteron wave functions. If orthogonality would be applied to the zero energy state one obtains an explicit relation of δ_1 , δ_2 and ϵ with the scattering lengths α_0 , α_2 and α_{02} as a direct generalization to the coupled channel case the one channel singlet case given by Eq. (47). The explicit expressions are rather cumbersome and will not be written down here explicitly. The results are depicted in Fig. 9 for Set IV. We observe a clear improvement in the threshold region, in consonance with the low energy parameters of Table III and a moderate improvement over the OPE results in the intermediate energy region. This suggests that finite cut-off effects may also be built in the phase shifts as well as the low energy parameters.

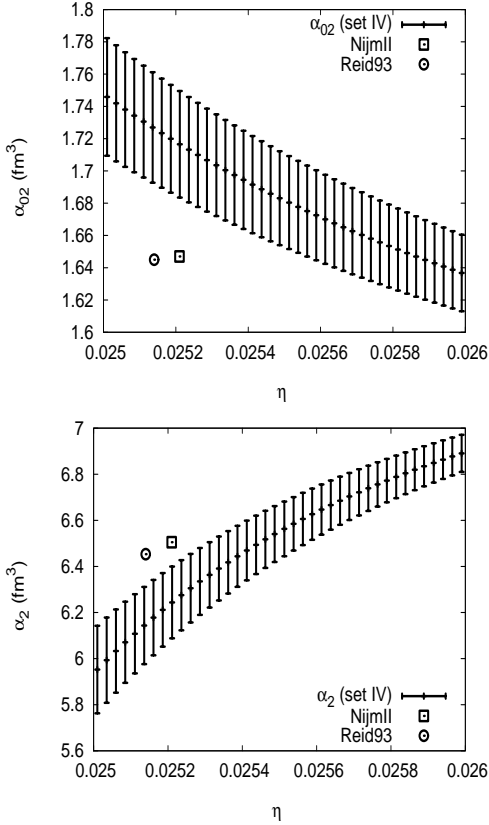


FIG. 8: The dependence of the α_{02} scattering length (in fm^3) and the α_2 scattering length (in fm^5) for the TPE potential as a function of the asymptotic D/S ratio η . The boxes represent the Reid93 and Nijm II values determined in Ref. [57]. We use $\alpha_0 = 5.419(7)\text{fm}$ to generate the bands. We use the set IV low energy constants.

VI. ERROR ANALYSIS AND DETERMINATION OF CHIRAL COUPLINGS FROM LOW ENERGY NN DATA AND THE DEUTERON

A. Propagating experimental errors in c_1 , c_3 and c_4 .

The results in the previous sections clearly show that deuteron and low energy scattering properties in the 1S_0 and $^3S_1 - ^3D_1$ channels are sensitive to finite cut-off effects and also to the values of the chiral constants after removal of the cut-off. We will assume that the values for c_1 , c_3 and c_4 are free of uncertainties. Then, Set IV provides the best description of triplet data but produces a slightly off value for the effective range in the singlet channel at the 2σ confidence level. Note that in the singlet case the theoretical prediction for r_0 does not have a large source of error as in the triplet case where uncertainties in η dominate the error. On the contrary, Set III provides a compatible value for the effective range in the singlet channel but incompatible values for the triplet

TABLE IV: Singlet 1S_0 and triplet $^3S_1 - ^3D_1$ scattering and deuteron properties with error estimates using the chiral TPE potential. We make a MonteCarlo calculation of the input parameters $g_{\pi NN} = 13.1 \pm 0.1$, $\alpha_{0,s} = -23.77 \pm 0.05$, $\alpha_{0,t} = 5.419 \pm 0.007$, $\eta = 0.0256 \pm 0.0004$ and the chiral constants c_1 , c_3 and c_4 . The quoted values span an interval where 68% of the output is contained.

	Set I	Set II	Set IV	Exp.
c_1	-0.81(15)	-0.76(7)	-0.81	
c_3	-4.69(1.34)	-5.08(24)	-3.20(16)	
c_4	3.40(4)	4.78(10)	5.40(1.65)	
$r_{0,s}$	$2.92^{+0.08}_{-0.04}$	$2.97^{+0.03}_{-0.02}$	$2.86^{+0.04}_{-0.03}$	2.77 ± 0.05
$r_{0,t}$	$1.36^{+0.33}_{-0.75}$	$1.48^{+0.14}_{-0.25}$	$1.76^{+0.03}_{-0.06}$	1.753 ± 0.008
A_s	$0.899^{+0.008}_{-0.009}$	$0.900^{+0.003}_{-0.004}$	$0.884^{+0.005}_{-0.008}$	0.8849 ± 0.0009
Q_d	$0.284^{+0.005}_{-0.007}$	$0.284^{+0.005}_{-0.004}$	$0.276^{+0.004}_{-0.004}$	0.2859 ± 0.0003
r_m	$1.998^{+0.015}_{-0.019}$	$1.998^{+0.007}_{-0.007}$	$1.965^{+0.011}_{-0.014}$	1.971 ± 0.006
P_d	$6.6^{+1.0}_{-0.9}$	$7.1^{+0.9}_{-0.9}$	$8.3^{+1.4}_{-1.5}$	—
α_{02}	$2.26^{+0.51}_{-0.39}$	$2.20^{+0.23}_{-0.16}$	$1.67^{+0.13}_{-0.13}$	—
α_2	$3.6^{+3.3}_{-6.9}$	$4.0^{+1.6}_{-2.9}$	$6.71^{+0.48}_{-0.83}$	—

channel in A_s and r_0 at the $3 - 4\sigma$ confidence level on the experimental side. Thus, on this basis we may reject Set III and accept Set IV.

To improve on this analysis, let us try to include some errors on the chiral coefficients. The πN analysis of Ref. [31] (Set I) and the NN fit of Ref.[11] (Set II) yield some errors. Ref. [28] (Set IV) does not quote errors but we will take the educated guess of a 5% error for c_3 and a 30% error for c_4 [58]. We can propagate then by a Monte-Carlo simulation implementing also the errors in $g_{\pi NN} = 13.1 \pm 0.1$, $\alpha_{0,s} = -23.77 \pm 0.05$, $\alpha_{0,t} = 5.419 \pm 0.007$, $\eta = 0.0256 \pm 0.0004$. We assume for simplicity that all these quantities are fully uncorrelated. This will in general enhance the errors, as compared to the case where correlations in c_1 , c_3 and c_4 with πN would be taken into account. Perhaps, the best thing would be to consider a simultaneous analysis of both NN and πN low energy data to build in correlations. Obviously, we do not expect good central values for the observables judging from Table III. But there is still the possibility of large error bars.

The outcoming distributions in the low energy and deuteron parameters are somewhat asymmetric. Actually, for a given set of c_1 , c_3 and c_4 distributions we observe the appearance of upper bounds in the 3S_1 effective range, namely $r_{0,t} \leq 1.79, 1.75, 1.81\text{fm}$ for Sets I,II and IV respectively where the out-coming distributions become more dense. The results of the error propagation are summarized in Table IV. Thus, we see that the values of the chiral coefficients deduced from low energy πN [31] are globally inconsistent, at the 1σ level, with the low energy NN threshold parameters after uncertainties are taken into account. The same remark applies to Set II [11]. Again, the loss of predictive power be-

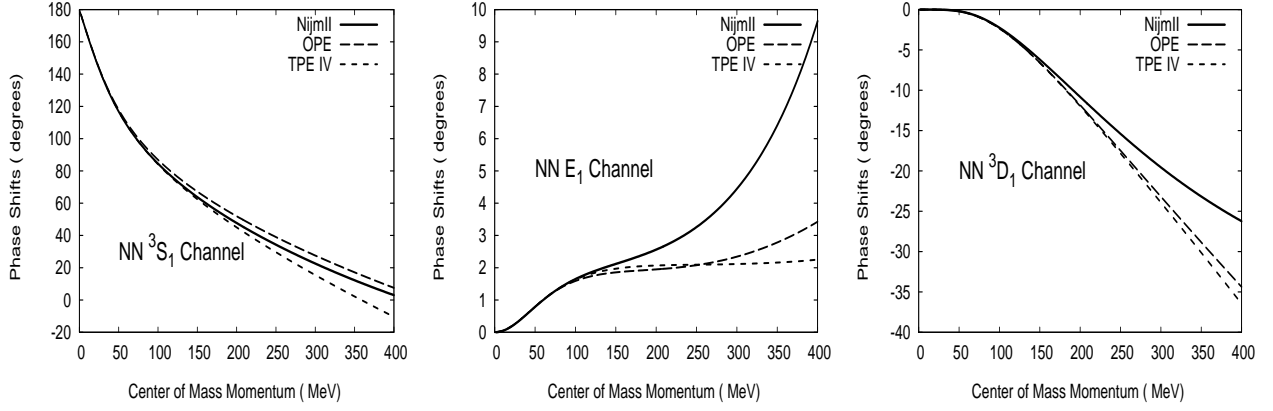


FIG. 9: Renormalized Eigen Phase shifts for the OPE and TPE potentials as a function of the CM np momentum in the triplet ${}^3S_1 - {}^3D_1$ channel compared to the Nijmegen results [43]. The regular scattering wave functions are orthogonal to the regular deuteron bound state wave functions constructed from the OPE with $\gamma = 0.231605\text{fm}^{-1}$, $m = 138.03\text{MeV}$ and $g_{\pi NN} = 13.083$ for the OPE contribution to the TPE potential and $g = 1.26$ for the TPE contribution to the TPE potential. We take Set IV (see main text).

comes manifest for all the sets although Set IV provides the best central values and the smallest errors. The situation for the quadrupole moment is noteworthy since the difference to the potential value is attributed to Meson Exchange Currents (MEC) and relativistic effects, which provide a correction of about 0.01fm^2 (see Ref.[25] in Ref. [19] and also Ref. [59]). As we see, this is about the size of the error deduced from our analysis. In the case of the deuteron matter radius the situation is even worse since MEC's contributions are much smaller 0.003fm [59] while our predicted errors are larger. It would be extremely interesting to reanalyze the problem with the present deuteron wave functions [60].

B. Determination of c_1 , c_3 and c_4 .

Another possibility is to attempt a direct fit to the data. The standard approach is to fit the partial waves to a NN database [43, 44]. The problem with such an approach is that, unfortunately, there is no error assignment on the phase shifts and hence a reliable assessment of errors cannot be made. Actually, besides the work of Ref. [11, 21] where a full partial wave analysis was undertaken, other works [23, 28, 29] assume fixed values for c_1 , c_3 and c_4 without attempting any error analysis based on input uncertainties. But even if data for the phase shifts with errors were known one expects the quality of the fit to worsen as the energy is increased as we think that the chiral approach to NN interaction should work best at low energies. If the data were known with uniform uncertainty one would fit until χ^2/DOF exceeds one providing an energy window. In such a fit all points are equally weighted while we know that the description at low energies, where the theory works best, will be compromised by the highest possible energy within such an

energy window.

Along the previous line of reasoning we propose, instead, to fit directly the low energy threshold parameters which central values and errors are well known and widely accepted. The basic ingredient is to use purely hadronic information in the process to avoid any contamination due to electromagnetic effects. Specifically, we make a Monte-Carlo sampling of the input data assuming that as primary data they are gaussian distributed and uncorrelated. For any of the samples we make a χ^2 fit to the values $r_{0,s}$, $r_{0,t}$ and A_S , i.e. we minimize

$$\chi^2 = \left(\frac{r_{0,s} - r_{0,s}^{\text{exp}}}{\Delta r_{0,s}} \right)^2 + \left(\frac{r_{0,t} - r_{0,t}^{\text{exp}}}{\Delta r_{0,t}} \right)^2 + \left(\frac{A_S - A_S^{\text{exp}}}{\Delta A_S} \right)^2 \quad (77)$$

and determine then the optimal values of c_1 , c_3 and c_4 . We only accept values where $\chi^2 < 1$, and the resulting distribution of chiral constants c_1 , c_3 and c_4 is given in Figs. 10. As we see, there is a very strong, almost linear, correlation between c_3 and c_4 . This can be easily understood in terms of the short distance dominance of the singlet effective range, since for the pure Van der Waals contribution, and in the limit of large scattering length $r_{0,s} \sim R = (MC_6)^{1/4}$, with C_6 given in Eq. (34). Deviations from linearity are induced from the larger relative error of $r_{0,s}$ (1%) as compared to $r_{0,t}$ and A_S (0.1%). This is different from the large scale partial phase-shift analysis obtained in Ref. [21] where a very small correlation between c_3 and c_4 of about 0.2 was found. We have checked that cutting-off data with decreasing values of χ^2 , excludes the points where the distribution is sparse, so that the dense part indeed reflects the uncertainties in the input data. The fact that the three coefficients seem to be on a line is just a consequence of solving by minimization a system of three equations and three un-

knowns.

We use $g_{\pi NN} = 13.083$, $\alpha_{0,s} = -23.77 \pm 0.05$, $\alpha_{0,t} = 5.419 \pm 0.007$, $\eta = 0.0256 \pm 0.0004$ and fit c_1 , c_3 and c_4 to the values $r_{0,s} = 2.77 \pm 0.05$, $r_{0,t} = 1.753 \pm 0.08$ and $A_S = 0.8846 \pm 0.0009$. Our final result for a sample with 125 points with $\chi^2 < 1$ is

$$\begin{aligned} c_1 &= -1.13_{-0.04}^{+0.02}(\text{stat}) \text{ GeV}^{-1}, \\ c_3 &= -2.60_{-0.23}^{+0.18}(\text{stat}) \text{ GeV}^{-1}, \\ c_4 &= +3.40_{-0.40}^{+0.25}(\text{stat}) \text{ GeV}^{-1}. \end{aligned} \quad (78)$$

The central value is the mean and the errors have been obtained by the standard method of excluding the 16% left and right extreme values of the variables, so as to have 68% confidence level between the upper and lower values. Cutting-off data with $\chi^2 < 0.5$ does not change significantly the result.

At the 2σ level, our values for c_1 , c_3 and c_4 are compatible with the analysis of low energy πN scattering of Ref. [31], $c_1 = -0.81 \pm 0.15$, $c_3 = -4.69 \pm 1.34$ and $c_4 = 3.40 \pm 0.04$, but incompatible with the NN full partial wave analyses [11, 21] where an energy dependent boundary condition at $a = 1.4\text{fm}$ was used. It is difficult to say whether other determinations for the chiral couplings based on NN scattering are incompatible with ours, since no error estimates have been provided.

C. Estimate of the systematic errors

As we have mentioned, any approach based on power counting of the potential cannot make an *a priori* estimate of the accuracy of the calculation. Nevertheless, we can have an idea by simply varying the input parameters.

At LO we may use either $g_A = 1.26$ as input or $g_{\pi NN} = 13.1$, since the difference is the Goldberger-Treiman discrepancy, which should be a higher order correction. The effect can be seen by comparing OPE with OPE* in Table III. When compared to the TPE result, for e.g. Set IV, the error is underestimated this way. At NNLO we use the same procedure in the TPE piece. Again, the difference should be higher orders. Numerically this is equivalent to include $g_{\pi NN} = 13.1 \pm 0.1$ and consider this systematic error as an statistical one, which has already been taken into account.

We can estimate the systematic error in the chiral constants by varying the input used to determine the c 's. To correlate the singlet and triplet channels we must keep $r_{0,s}$ and $\alpha_{0,s}$. So we can interchange the inputs A_S , $r_{0,t}$ with the outputs r_m , Q_d , α_{02} and α_2 . This yields a total of 15 possible combinations. Another question concerns the assessment an error to the fitted variables whenever there is no direct experimental quantity, since this choice weights the determination of the c 's. This is the case for α_{02} and α_2 , where we make the educated guess of taking the difference between the Reid93 and NijmII values as determined in Ref. [57] as an estimate of the error. The

situation with Q_d is a bit special and we exclude it from the analysis¹³. The results are listed in Table V. We see that this estimate on the systematic error provides a larger fluctuation than direct propagation of the input errors for c_1 . Symmetrizing the errors we get

$$\begin{aligned} c_1 &= -1.2 \pm 0.2 (\text{syst}) \text{ GeV}^{-1}, \\ c_3 &= -2.6 \pm 0.1 (\text{syst}) \text{ GeV}^{-1}, \\ c_4 &= +3.3 \pm 0.1 (\text{syst}) \text{ GeV}^{-1}. \end{aligned} \quad (79)$$

If we attempt a fit to all observables assigning $\Delta Q_d = 0.01\text{fm}^2$ and $\Delta\alpha_{02} = 0.4\text{fm}$ we get $c_1 = -0.9$, $c_3 = -2.71$ and $c_4 = 3.85$ with a large $\chi^2/DOF = 3$ basically due to the small errors. Obviously a more realistic estimate of the errors would be desirable.

VII. THE ROLE OF CHIRAL VAN DER WAALS FORCES

As we have pointed out, our approach is not the conventional one of adding short distance counterterms following a given *a priori* power counting regardless on the approximation where the long distance potential has been constructed. Instead, the potential power counting dictates the form of the short distance physics by demanding a finite limit when the regulator is removed. In order to stress the differences with previous approaches it is interesting to see how much of the phase shifts is determined from the short distance chiral potential *without* adding a short range contribution to the effective range. In the standard approach this can be achieved by adding a counterterm C_2 in the S -wave channels. In Ref. [41] we showed that both perturbatively and non-perturbatively the orthogonality constraints for the OPE potential imply $C_2 = 0$. Here we will see that the bulk of the S -wave interaction can be explained mainly in terms of the chiral Van der Waals force when renormalization is carried

TABLE V: Central values for the chiral constants c_1 , c_3 and c_4 depending on the input. we only include $\chi^2 < 1$.

Fitted	c_1	c_3	c_4	χ^2
$r_{0,s}, A_S, r_{0,t}$	-1.09	-2.61	3.36	0.06
$r_{0,s}, A_S, r_m$	-1.23	-2.57	3.34	0.3
$r_{0,s}, r_{0,t}, r_m$	-1.45	-2.54	3.26	0.2
$r_{0,s}, r_{0,t}, \alpha_2$	-1.09	-2.64	3.17	0.4
$r_{0,s}, r_m, \alpha_2$	-1.03	-2.70	3.26	0.03

¹³ The discrepancy of potential models to the experimental value $\sim 0.01\text{fm}^2$, attributed to MEC's and relativistic effects [59], is about two orders of magnitude larger than the error in the experimental number $\sim 0.0003\text{fm}^2$ and the discrepancy between potential models $\sim 0.0004\text{fm}^2$. It is not clear whether the discrepancy can be pinned down with similar errors [60].

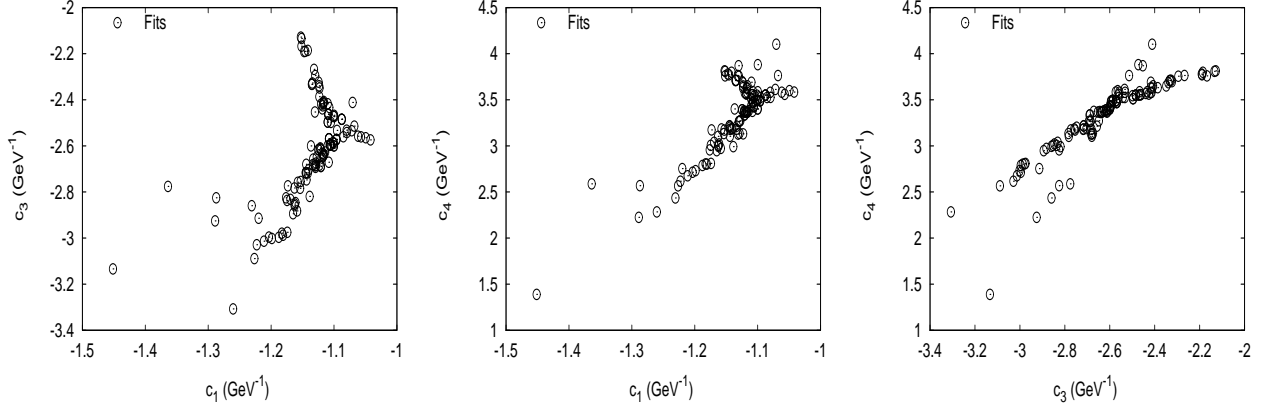


FIG. 10: Correlation plots for the low energy constants c_1 , c_3 and c_4 obtained from the renormalized TPE potential by fitting the singlet and triplet effective ranges $r_{0,s} = 2.77 \pm 0.05$ and $r_{0,t} = 1.753 \pm 0.08$ and the deuteron s-wave asymptotic normalization $A_S = 0.8846 \pm 0.0009$. The dispersion in the data reflects the dispersion in all input parameters, $g_{\pi NN} = 13.11 \pm 0.08$, $\alpha_{0,s} = -23.77 \pm 0.05$, $\alpha_{0,t} = 5.419 \pm 0.007$, $\eta = 0.0256 \pm 0.0004$ which provide $\chi^2 < 1$.

out, without any additional short distance contribution or counterterm.

For a pure Van der Waals potential of the form

$$U = -\frac{R^4}{r^6}, \quad (80)$$

the zero energy wave function can be analytically computed [47] in terms of Bessel functions $J_\nu(x)$. Normalizing to the asymptotic form $u_0(r) \rightarrow 1 - r/\alpha_0$ we get

$$u_0(r) = \Gamma\left(\frac{5}{4}\right) \sqrt{\frac{2r}{R}} J_{\frac{1}{4}}\left(\frac{R^2}{2r^2}\right) - \Gamma\left(\frac{3}{4}\right) \sqrt{\frac{rR}{2}} J_{-\frac{1}{4}}\left(\frac{R^2}{2r^2}\right) \frac{1}{\alpha_0}. \quad (81)$$

The effective range can also be computed analytically [61, 62] from Eq. (37) yielding

$$r_0 = \frac{-4R^2}{3\alpha_0} + \frac{4R^3\Gamma(\frac{3}{4})^2}{3\alpha_0^2\pi} + \frac{16R\Gamma(\frac{5}{4})^2}{3\pi}, \\ = 1.39473R - \frac{1.33333R^2}{\alpha_0} + \frac{0.637318R^3}{\alpha_0^2}, \quad (82)$$

in agreement with the general low energy theorem of Eq. (39). Taking the values of Table I for $R = (MC_6)^{1/4}$ one gets in the singlet 1S_0 channel

$$r_{0,s} = 2.39811 - \frac{3.9418}{\alpha_{0,s}} + \frac{3.23959}{\alpha_{0,s}^2} \quad (\text{Set I}), \\ r_{0,s} = 2.49192 - \frac{4.25624}{\alpha_{0,s}} + \frac{3.63486}{\alpha_{0,s}^2} \quad (\text{Set II}), \\ r_{0,s} = 2.2227 - \frac{3.8625}{\alpha_{0,s}} + \frac{2.57944}{\alpha_{0,s}^2} \quad (\text{Set III}), \\ r_{0,s} = 2.29099 - \frac{3.59753}{\alpha_{0,s}} + \frac{2.82459}{\alpha_{0,s}^2} \quad (\text{Set IV}). \quad (83)$$

The numerical agreement at the ten percent level of the $\alpha_{0,s}$ independent term with the full chiral TPE result, Eq. (43)), is striking¹⁴. On the other hand, first order perturbation theory in the OPE potential yields (see Sect. A of Ref. [41]) in the form of Eq. (39) the result

$$r_{0,s} = \frac{g_{\pi NN}^2}{8M\pi} \left(1 - \frac{8}{3\alpha_{0,s}m} + \frac{2}{\alpha_{0,s}^2m^2}\right), \\ = 1.4369 - \frac{5.4789}{\alpha_{0,s}} + \frac{5.8758}{\alpha_{0,s}^2}. \quad (84)$$

Note that the coefficient in $1/\alpha_{0,s}^2$ is slightly better described by the OPE perturbative value than the full OPE result (see Eq. (43)), a not unreasonable result since this coefficient is sensitive to the longest range part of the interaction. Likewise, the bulk of the α_0 -independent coefficient is given *just* by the most singular contribution to the full chiral potential. As we see, for large scattering lengths the effective range scales with the Van der Waals singlet radius $R_s = (MC_{6,^1S_0})^{1/4}$ and not with the pion Compton wavelength $1/m$, confirming the dominance of the short distances singularity in the singlet channel.

For the triplet channel, the equation cannot be solved analytically, and the effective range has a correction due to the D-wave (see Eq. (67)). Moreover, the scattering length is a factor five times smaller than in the singlet case, so that we do not expect in principle such a dramatic agreement. If we neglect the mixing with the D-wave and take the $R_t = (MC_{6,^3S_1})^{1/4}$ of Eq. (52) we

¹⁴ The formula (82) can also be used as a numerical test of the integration method and of the numerical solution of the differential equations. This is a non-trivial condition due to the rapid oscillations of the wave function at the origin. We have checked that it is accurately reproduced.

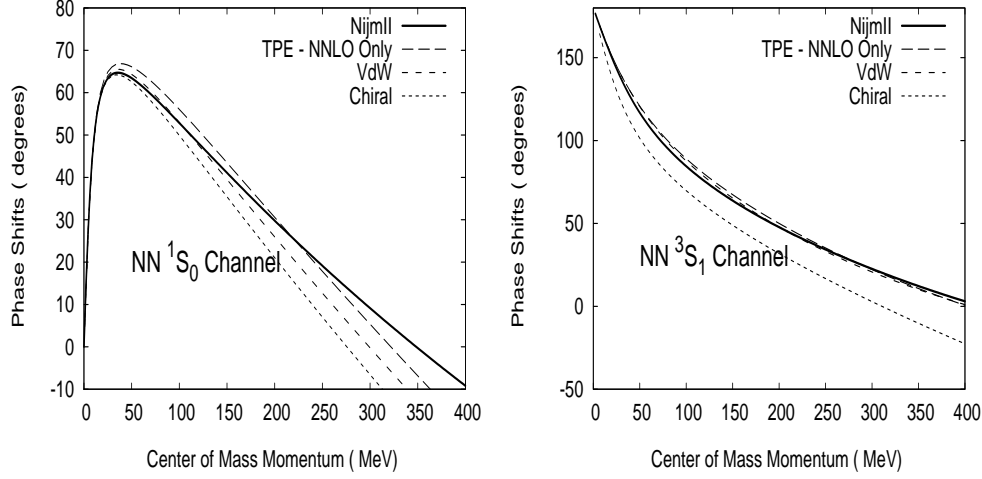


FIG. 11: Renormalized Eigen Phase shifts in the 1S_0 and $^3S_1 - ^3D_1$ channel for the pure chiral Van der Waals C_6/r^6 potential (Vdw), and the pure NNLO terms compared to the renormalized phase shifts with the same parameters from Table I for Set IV. We also compare to the Nijmegen database [43].

get

$$\begin{aligned}
 r_{0,t} &= 2.50174 - \frac{4.28983}{\alpha_{0,t}} + \frac{3.67797}{\alpha_{0,t}^2} & (\text{Set I}), \\
 r_{0,t} &= 2.58537 - \frac{4.58143}{\alpha_{0,t}} + \frac{4.05928}{\alpha_{0,t}^2} & (\text{Set II}), \\
 r_{0,t} &= 2.35089 - \frac{3.78809}{\alpha_{0,t}} + \frac{3.05196}{\alpha_{0,t}^2} & (\text{Set III}), \\
 r_{0,t} &= 2.40877 - \frac{3.97691}{\alpha_{0,t}} + \frac{3.28297}{\alpha_{0,t}^2} & (\text{Set IV}),
 \end{aligned} \tag{85}$$

which, using the triplet scattering length value, $\alpha_{0,t} = 5.42$ yields $r_{0,t} = 1.83, 1.87, 1.75, 1.78$ respectively, in remarkable agreement with the experimental value. An estimate of the mixing effect can be made by using the largest van der Waals eigen radius $R_+ = (MC_{6,+})^{\frac{1}{4}}$ obtained by diagonalizing the interaction at short distances. From Table I, Eq. (82) and the experimental value of the scattering length we get $r_0 = 2.00, 2.07, 1.95, 2.05\text{fm}$ for Sets I,II,III and IV respectively, accounting for about 85% percent of the full value. Instead, perturbation theory for OPE, Eq. (84) yields $r_0 = 0.62\text{fm}$, and full OPE $r_0 = 1.64$. Actually, using the relation

$$\begin{aligned}
 MC_{6,^1S_0} - MC_{6,^3S_1} &= R_s^4 - R_t^4 \\
 &= \frac{3g^2}{64\pi^2 f^4} (4 - 9g^2) \tag{86}
 \end{aligned}$$

we get an explicit correlation between $\alpha_{0,s}$, $r_{0,s}$, $\alpha_{0,t}$ and $r_{0,t}$ regardless on the numerical values of the chiral constants c_3 and c_4 . In the range of physical parameters this looks like a linear correlation (see Fig. 12) between the singlet and triplet effective ranges. For $r_{0,t} = 1.75$ one gets $r_{0,s} = 2.34$.

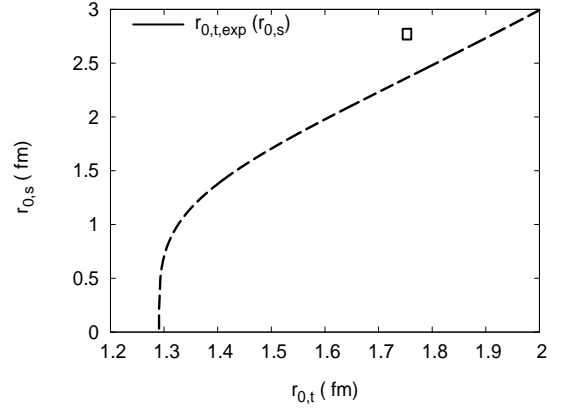


FIG. 12: Van der Waals correlation between the singlet and triplet effective ranges using the experimental singlet and triplet scattering lengths. The point represents the experimental values.

To check further the dominance of chiral Van der Waals interactions, we plot in Fig. 11 the phase shifts for a variety of situations including the pure Van der Waals contributions, as well as the contribution of the NNLO only, which reduces to the previous case at short distances but decays exponentially as $\sim e^{-2mr}$ at long distances. The plots confirm, again, our estimations based on the pure Van der Waals potential of the effective range for the s -waves, and this is the reason why the triplet s -wave is better reproduced than the singlet case for Set IV. Obviously, by adjusting the effective range changing the chiral parameters c_3 and c_4 we could obtain a much better description of the data.

The results of this study show that the singularity of

the chiral Van der Waals force is not a feature that should be avoided, but instead provides a very simple way to describe the scattering data for the s -waves.

Finally, it is interesting to note, that central waves based on taking the chiral limit of the potential are less accurately described than the phase-shifts obtained from the pure Van der Waals contribution. In this limit the singlet 1S_0 channel contains in addition to the Van der Waals term a $1/r^5$ contribution stemming from the NLO TPE contribution. The triplet $^3S_1 - ^3D_1$ TPE contribution has a similar structure in addition to the OPE tensor $1/r^3$ singular short distance contribution.

VIII. THE TPE POTENTIAL AT NLO: A MISSING LINK ?

In the previous sections we analyzed the renormalization of the NNLO potential. In this Section we analyze the NLO in the singlet 1S_0 and triplet $^3S_1 - ^3D_1$ channels and the problem that arises in the latter. We argue that similar trends are observed in finite cut-off calculations. We also suggest several scenarios on how the problem may be overcome.

A. Convergence in the singlet 1S_0 channel

In the singlet 1S_0 channel the potential at short distances behaves as [7, 11, 12]

$$U_{^1S_0} \rightarrow \frac{MC_{5,^1S_0}}{r^5}, \quad (87)$$

where

$$MC_{5,^1S_0} = \frac{M(1 + 10g^2 - 59g^4)}{256\pi^3 f^4} \quad (88)$$

The singlet coefficient is negative and, according to the discussion in Sect. III, one has an undetermined short distance phase which can be fixed by using the scattering length as input. The effective range in the singlet channel is given by

$$r_0 = 2.122 - \frac{4.889}{\alpha_0} + \frac{5.499}{\alpha_0^2} \quad (\text{NLO}), \quad (89)$$

which compared with the LO and NNLO results, Eq. (43), shows a convergence rate. To show that this trend to convergence is not fortuitous we display in Table VI the threshold parameters of the effective range expansion $k \cot \delta = -1/\alpha_0 + r_0 k^2/2 + v_2 k^4 + v_3 k^6 + v_4 k^8$ depending on the terms kept in the expansion of the potential given by Eq. (1). As we see there is a clear trend to convergence, although the higher order threshold parameters display a slower convergence rate since they are increasingly sensitive to the shorter range regions. This trend is confirmed in Fig. 13 for the phase shift. Obviously, there is scale separation in the singlet potential, and higher order potentials although more singular at the origin yield contributions in the right direction.

B. The problem in the triplet $^3S_1 - ^3D_1$ channel

The triplet $^3S_1 - ^3D_1$ potential at short distances has the behaviour [7, 11, 12]

$$\begin{aligned} U_{^3S_1}(r) &\rightarrow \frac{MC_{5,^3S_1}}{r^5}, \\ U_{E_1}(r) &\rightarrow \frac{MC_{5,E_1}}{r^5}, \\ U_{^3D_1}(r) &\rightarrow \frac{MC_{5,^3D_1}}{r^5}, \end{aligned} \quad (90)$$

where

$$\begin{aligned} MC_{5,^3S_1} &= \frac{3M(-1 - 10g^2 + 27g^4)}{256\pi^3 f^4}, \\ MC_{5,E_1} &= -\frac{15Mg^4}{64\sqrt{2}\pi^3 f^4} \\ MC_{5,^3D_1} &= \frac{3M(-1 - 10g^2 + 37g^4)}{256\pi^3 f^4}, \end{aligned} \quad (91)$$

On the other hand, the diagonalized triplet coefficients are

$$\begin{aligned} MC_{5,+} &= \frac{3M(-1 - 10g^2 + 17g^4)}{256\pi^3 f^4}, \\ MC_{5,-} &= \frac{3M(-1 - 10g^2 + 47g^4)}{256\pi^3 f^4}, \end{aligned} \quad (92)$$

and the mixing angle is given by $\tan \theta = \sqrt{2}$, differing by $-\pi$ as compared to the OPE case [41]. For $0.5356 < g < 0.8217$ one would have an attractive-repulsive situation (see Sect. III), as in the OPE case [41] and in such a case one could take either the deuteron binding energy or the 3S_1 scattering length. However, for the physical value $g = 1.26$ one has two short distance repulsive eigenchannels, and hence one must take the exponentially decaying regular solutions at the origin. Let us remind the according to Sect. III *finite renormalized results can only be obtained by precisely choosing*

TABLE VI: Convergence of the threshold parameters of the effective range expansion $k \cot \delta = -1/\alpha_0 + r_0 k^2/2 + v_2 k^4 + v_3 k^6 + v_4 k^8$ in the singlet 1S_0 channel depending on the successive inclusion of terms in the potential $U = U_{\text{LO}} + U_{\text{NLO}} + U_{\text{NNLO}} + \dots$. LO means LO alone (and taking $g_{\pi NN} = 13.083$ and $g_A = 1.26$), NLO means LO+NLO and so on. The only input is the scattering length α_0 besides the potential parameters. For the NNLO case we use Set IV for the chiral constants c_1 , c_3 and c_4 given in Table I.

1S_0	LO	NLO	NNLO	Exp.	Nijm II
$\alpha_0(\text{fm})$	Input	Input	Input	-23.74(2)	-23.73
$r_0(\text{fm})$	1.44	2.29	2.86	2.77(5)	2.67
$v_2(\text{fm}^3)$	-2.11	-1.02	-0.36	—	-0.48
$v_3(\text{fm}^5)$	9.48	6.09	4.86	—	3.96
$v_4(\text{fm}^7)$	-51.31	-35.16	-27.64	—	-19.88

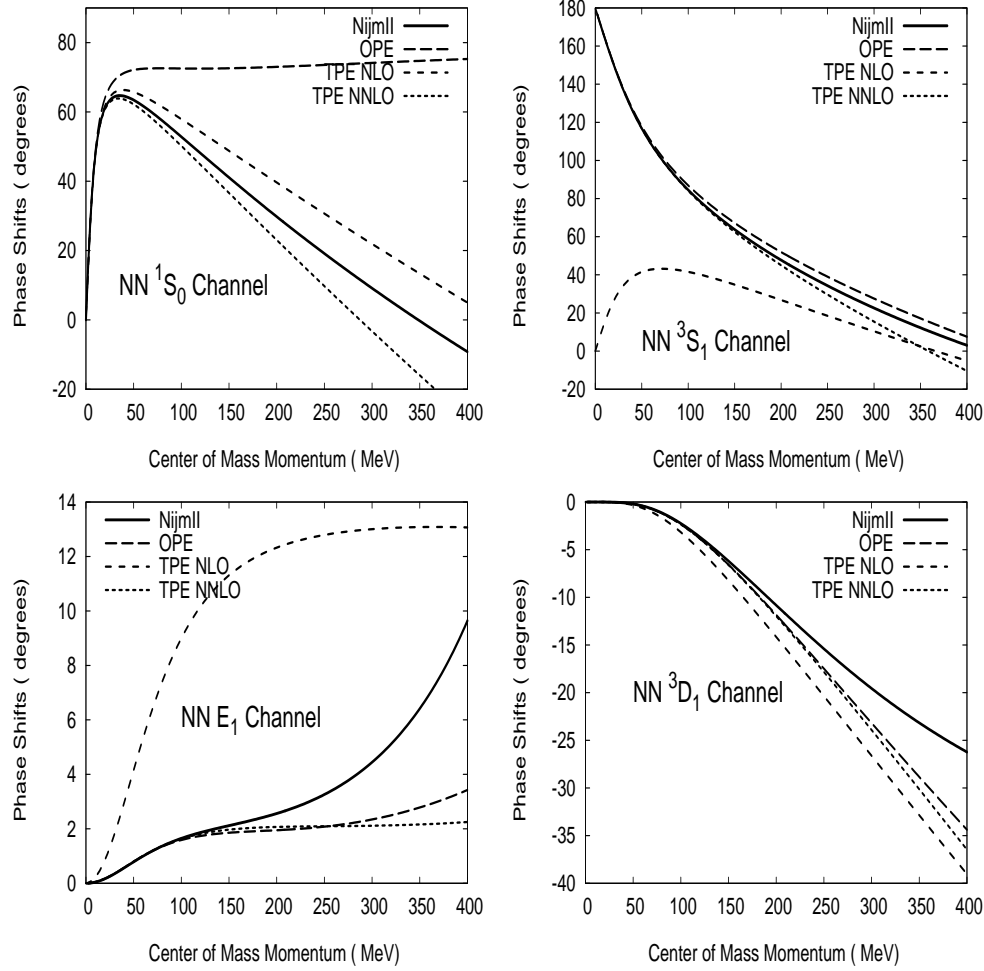


FIG. 13: Renormalized Eigen phase shifts at LO, NLO and NNLO as a function of the CM np momentum k in the singlet 1S_0 and triplet $^3S_1 - ^3D_1$ channels compared to the Nijmegen results [43] for different parameter sets.

the regular solution at the origin. In this case there are no short distance phases, and the scattering lengths, as well as the phase shifts are completely determined from the potential. The (finite) renormalized results are depicted in Fig. 13. As we see, the singlet 1S_0 phase-shift shows a very reasonable trend, since NLO improves on the LO, and it is improved by the NNLO potential. We remind that in the three cases the scattering length is exactly the same. However, not completely unexpectedly, the triplet channel results worsen the LO ones. In the next subsection we show that if, demanding the standard Weinberg counting requires the irregular solution at the origin, hence yielding to divergent renormalized results.

C. Finite cut-offs and the Weinberg counting

The special status of the NLO calculation as compared to the LO and NNLO ones has been recognized in previ-

ous studies in momentum space [10] where regularization was implemented by using a sharp cut-off Λ . As noted by these authors, the allowed cut-off variations at NLO are *smaller* ($\sim 380 - 600$ MeV) than at LO ($\sim 700 - 800$ MeV) or NNLO ($\sim 800 - 1000$) but the reasons have not been made clear. Let us focus on the triplet $^3S_1 - ^3D_1$ channel. Within our coordinate space renormalization scheme this trend can be easily understood. At LO one fixes only one parameter, say α_0 , and because one has attractive and repulsive potentials at short distances the system will naturally be driven into the exponential regular solution at the origin. Obviously, if one would fix some other parameter independently, say r_0 (or equivalently using a counterterm C_2 , and not the one predicted by the regular solutions, one would be driven instead to the irregular solution, not allowing to remove the cut-off in practice. In such a situation one would be forced to keep the cut-off finite at the scale where the repulsive core sets in. However, at LO the Weinberg power counting does not allow to fix this additional parameter and

one can comfortably reach higher cut-off values. On the contrary, at NLO one has two repulsive eigenpotentials and one cannot fix any low energy parameter arbitrarily. Otherwise one would be attracted to the irregular solutions at short distances. On the other hand, they are attractive at long distances, so that one would expect a stability region where the potential becomes flat before turning into a repulsive core in both eigenchannels. This is exactly what one observes in the NLO calculation of Ref. [10]. The occurrence of such a plateau is to some extent fortuitous since it is associated to the critical points of the potential, and not with some a priori estimate on the validity range of the NLO potential. Finally, in the NNLO calculation because both eigenpotentials have an attractive character one can again increase the cut-off since there are no irregular solutions in the problem where one can be attracted to. It is very rewarding that our coordinate space analysis of short distance singularities anticipates when these features can be expected. On the other hand this does not imply that finite cut-off calculations are necessarily wrong, simply that the observed features when the cut-off approaches the limit can be understood.

The previous discussion can be illustrated in our approach by looking at the short distance cut-off dependence of the effective range, r_0 (in fm) and the deuteron wave function renormalization, A_S (in $\text{fm}^{-1/2}$), in the triplet $^3S_1 - ^3D_1$ channel at LO, NLO and NNLO in the standard Weinberg counting as presented in Fig. 14. As described in Sect. III C any counterterms can be mapped into a given renormalization condition. Once these conditions are fixed we can ask whether other properties are finite or not. Thus, at LO we fix the deuteron binding energy (one counterterm), at NLO and NNLO we fix the deuteron binding energy, the asymptotic D/S -mixing, η , and the scattering length α_0 (three counterterms). In all cases it is clear that by lowering the cut-off at LO and NNLO of the approximation one nicely approaches the experimental values. This raises immediately the question whether there is a given value of the cut-off where NLO improves over LO. As we see such a region does not exist. In addition, although there is a nice and clear trend in both LO and NNLO for distances below 0.5fm for A_S down to the origin, this is not so at NLO. So, in this case it is not true that low energy properties are independent on short distance details, in contrast to the standard EFT wisdom. Moreover, Fig. 14 shows explicitly the conflict between the Weinberg counting and the remotion of the cut-off at NLO because r_0 and A_S diverge due to the onset of the irregular solution, as anticipated in our study of short distance solutions (see Sect. III). We have checked that this is a general feature on both deuteron and scattering properties. On the contrary, LO and NNLO have a rather smooth limit because in these two cases Weinberg power counting on the short distance counterterms turns out to be compatible with the choice of the regular solution at the origin. Thus, *in the $^3S_1 - ^3D_1$ channel the renormalized solution at NLO in the Weinberg counting*

is divergent while LO and NNLO are convergent. In conclusion, the present analysis shows in a somewhat complementary manner as done in Sect. VIII B that indeed the NLO is problematic, at least non-perturbatively. In Sect. IX we will see that the problem is not solved if the NLO contribution is computed within a perturbative framework using the exact OPE-distorted wave basis as a lowest order approximation.

D. The role of relativity and the Δ resonance in the renormalization problem

The requirement of renormalizability may be regarded as a radical step, and renormalized LO calculations demand violating dimensional power counting on the counterterms [42] in non central waves such as 3P_0 , due to an attractive $1/r^3$ singularity. To reach a finite limit the authors of Ref. [42] propose to promote counterterms which in Weinberg's power counting are of higher order. However, in their proposal it is intriguing that they choose to promote just one counterterm in coupled channels, while they could have used a coupled channel counterterm, i.e. three counterterms in total. In the boundary condition approach we know from the start how many independent parameters must be *exactly* taken to reach a finite and unique limit, the reference to power counting is only specified at the level of the potential. Note that the power counting in the potential fixes its short attractive-repulsive singular character, and this is the origin of the conflict of assuming an *a priori* power counting for the counterterms. Finiteness requires that some forbidden counterterms must be allowed (promoted) [42] but also that some allowed counterterms must be forbidden (demoted). In such a framework, our NLO calculations in the $^3S_1 - ^3D_1$ channel lead to finite but nonsensical results due to the repulsive-repulsive $1/r^5$ singularity (See Sect. VIII). On the other hand, if one fixes as required by the power counting the scattering length, the limit does not exist because one is driven to the exponentially diverging solution at the origin (For instance, Eq. (67) gives $r_0 \rightarrow -\infty$). How then can we reconcile finiteness with fixing of the parameters ?. As we pointed out already, the singular short distance behaviour of the chiral potential is in fact a long distance feature which changes dramatically when changing the long distance physics. Actually, one may reverse the argument and use renormalizability as a selective criterium for admissible long distance potentials. In the following we want to provide at least two possible scenarios how this might happen, i.e. , how modifying the potential at long distances by introducing physically relevant information the short distance behaviour of the potential changes.

In the first place, the chiral potential, Eq. (1), was derived in the heavy baryon expansion. The short distance character may change when not taking such a limit since the combination Mr does make the order of limits ambiguous. A proper treatment of relativistic effects

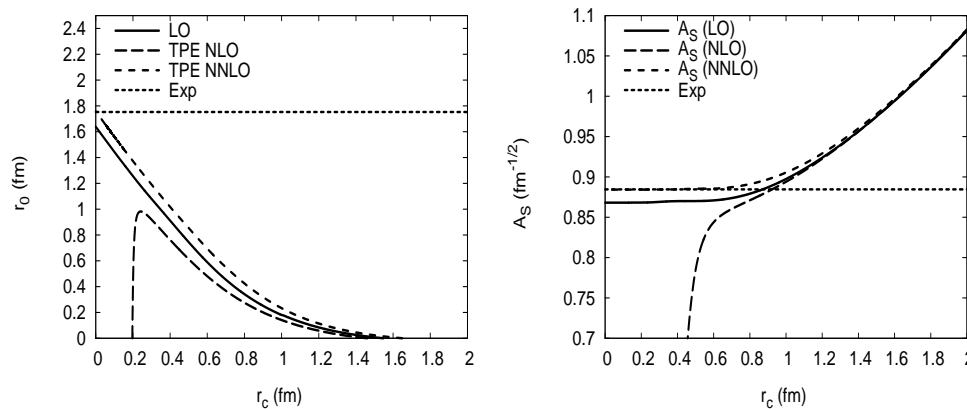


FIG. 14: Cut-off dependence of the effective range, r_0 (in fm) and the deuteron wave function renormalization, A_S (in $\text{fm}^{-1/2}$), in the triplet ${}^3S_1 - {}^3D_1$ channel at LO, NLO and NNLO in the Weinberg counting. At LO we fix the deuteron binding energy (one counterterm), at NLO and NNLO we fix the deuteron binding energy, the asymptotic D/S -mixing, η and the scattering length α_0 (three counterterms). We use Set IV of chiral coupling constants.

requires inclusion of antinucleons in loops, and a satisfactory EFT treatment of relativistic effects remains a challenging open problem because the standard crossing vs. unitarity non-perturbative divorce. On top of this one should use a satisfactory relativistic two body equation, which necessarily makes the problem fully non-local in coordinate space. Nevertheless, there exist “relativistic” potentials where *some* of the terms of higher power in $1/M$ than the TPE obtained in heavy-baryon ChPT are kept [25, 26, 27] which have $1/r^7$ Van der Waals short distance behaviour with attractive-repulsive eigen potentials [63] meaning that as in the OPE case one has one free parameter. A calculation using these incomplete “relativistic” potentials will be presented elsewhere [64].

A second scenario is related to the role played by the Δ resonance¹⁵ not included in the present analysis. As pointed out in Ref. [8], the Δ provides the bulk of the chiral constants, yielding $-c_3 = 2c_4 = g_A^2/2\Delta$, with $\Delta = 293\text{MeV}$ the nucleon-delta mass splitting, yielding $c_3 = -2.7\text{GeV}^{-1}$ and $c_4 = 1.35\text{GeV}^{-1}$. The difference to the parameters of Ref. [20] may be due to some other resonances. On the other hand, in terms of scales one has $\Delta \sim 2m_\pi$, which might be regarded as a small parameter. This obviously does not mean that Δ vanishes in the chiral limit. In the standard chiral counting of the potential, Eq. (1), the combinations $\bar{c}_1 = Mc_1$, $\bar{c}_3 = Mc_3$ and $\bar{c}_4 = Mc_4$ are considered to be zeroth order, but according to the previous argument they could be regarded to be enhanced by one negative power. Thus the nominally NNLO terms containing c_3 and c_4 might become NLO contributions, and hence changing the repulsive-repulsive $1/r^5$ singularity into an attractive-attractive $1/r^6$ one. On the other hand, the c_3 and c_4 contributions of the

standard NNLO dominate the short distance Van der Waals contributions. Actually, much of the NNLO potential is built from these terms all over the range. According to this reasoning our NNLO calculation may be closer to a NLO one where the $N\Delta$ splitting is regarded as small parameter. In fact, taking NLO+ Δ with $-c_3 = 2c_4 = g_A^2/2\Delta$ and $\eta = 0.0256$ one gets $A_S = 0.8869\text{fm}^{-1/2}$, $Q_D = 0.2762\text{fm}^2$, $r_m = 1.9726\text{fm}$ and $P_d = 0.06$ in overall agreement with Table III. It would be rather interesting to look for further consequences of this Δ -counting at higher orders. The importance of the Δ in the NN problem has been stressed in several works already on power counting grounds [2, 3, 10, 65] but the crucial role played on the renormalization problem, i.e., the fact that the cut-off can be completely removed has not been recognized. Our discussion suggests that the momentum space cut-off could also be comfortably removed in this Δ -counting, unlike the delta-less NLO.

The two possible scenarios outlined above do not prove that the requirement of renormalizability is necessarily right, but suggest that looking into the short distance singular behaviour of long distance chiral potentials together with the mathematical requirement of finiteness may provide a significant physical insight into the NN problem. In the language of Ref. [42] where promotion of counterterms on the basis of the renormalizability requirement has been stressed, we are perhaps led also to the demotion of counterterms (like for relativistic potentials), or alternatively the promotion of terms in the potential (like in the Δ counting described above).

E. Van der Waals forces, the molecular analogy and the chiral quark model

The previous arguments show that it is possible to change the attractive/repulsive character of the poten-

¹⁵ We thank D. Phillips for drawing our attention to this point.

tial at short distances by organizing the calculation of the potential in a different manner, but does not give a clue on why this actually happens. Remarkably, the analogy with atomic neutral systems subjected to Van der Waals forces illustrated in Sect. VII goes further and provides valuable insight into the problem. In low energy molecular physics where one works in a Born-Oppenheimer approximation, all atomic constituents, electrons and nuclei interact through the Coulomb force arising from one photon exchange. At long distances between distant electrons the potential is a dipole-dipole interaction

$$V_{\text{dip}}(R) = e^2 \sum_{A,B} \left[\frac{\vec{r}_A \cdot \vec{r}_B}{R^3} - 3 \frac{(\vec{r}_A \cdot \vec{R})(\vec{r}_B \cdot \vec{R})}{R^5} \right] \quad (93)$$

where the sum runs over electrons belonging to different atoms. In second order perturbation theory the atom-atom energy at a separation distance R reads,

$$V_{AA} = \langle AA | V_{\text{dip}} | AA \rangle + \sum_{AA \neq A^* A^*} \frac{|\langle AA | V_{\text{dip}} | A^* A^* \rangle|^2}{E_{AA} - E_{A^* A^*}} + \dots \quad (94)$$

where $|AA\rangle$ and $|A^* A^*\rangle$ is the electron wave function corresponding to a pair of separated clusters in their atomic ground state and excited states respectively. The first order contribution vanishes for atoms with no permanent dipole moment. The mutual electric polarization causes the Van der Waals interaction between the two atoms, C_6/R^6 and because it is second order perturbation theory it is obvious that the C_6 contribution to the potential will always be attractive. However, it is not clear that higher order terms would always be attractive. It is remarkable that the theorem of Thirring and Lieb [66] establishes that the Coulomb force between constituents implies all terms in the expansion being attractive, without appealing to the dipole-approximation. Thus, according to this result the long distance force will always be singular and attractive at short distances, and that is exactly what one needs. In such a situation making a long distance expansion of the potential, $U = -R_6^4/r^6 - R_8^6/r^8 + \dots$ and computing the scattering phase shifts by fixing always the same scattering length, along the lines pursued in this paper, makes much sense. Moreover, one expects the results for the phase shifts to be convergent if there is scale separation between the corresponding Van der Waals radii $R_6 \gg R_8 \gg \dots$. Our experience with several atomic systems confirms these expectations [67].

The argument in the NN system is a straightforward generalization of the molecular system above. It is well known that there are no colour hidden states between colour neutral systems, so that at long distances one may assume only exchange of colourless objects. The longest range object will be the pion, and the mutual (chiral) polarizability will cause attraction between the nucleons, exactly in the same way as for atom-atom interactions. If we use as an example the chiral quark model, assume for simplicity non-relativistic constituent quarks one obtains

the OPE for quarks. To second order perturbation theory we get the NN potential in the Born-Oppenheimer approximation

$$V_{NN} = \langle NN | V_{\text{OPE}} | NN \rangle + \sum_{HH \neq NN} \frac{|\langle NN | V_{\text{OPE}} | HH \rangle|^2}{E_{NN} - E_{HH}} + \dots \quad (95)$$

where V_{NN} represents the potential in the NN operator basis. This yields *exactly* when $HH = N\Delta$ the results found in Ref. [2, 3, 8] and naturally explains why the contribution from one Δ intermediate state is attractive at short distances. Although this analogy with molecular systems is very suggestive, the generalization to all orders along the lines of the Lieb-Thirring theorem within a QCD context remains at present an optimistic speculation.

IX. RENORMALIZED PERTURBATION THEORY VERSUS NON-INTEGGER POWER COUNTING

A. Perturbations on boundary conditions

In all our calculations we have taken a long distance potential calculated perturbatively, and scattering amplitudes have been computed non-perturbatively by fully iterating a potential computed in perturbation theory, as initially suggested by Weinberg [1]. We will call this form of solution non-perturbative for brevity. This requires a non-perturbative treatment of the renormalization problem, which naturally implies that the short distance renormalization conditions (or counterterms) are determined by the most singular contribution of the long distance potential at the origin. For the NN chiral potential it turns out that the higher the order the more singular the potential. As a consequence we have seen in Sect. VIII C that for instance Weinberg counting at NLO in the ${}^3S_1 - {}^3D_1$ channel is incompatible with renormalization and finiteness due to the short distance repulsive character of the NLO potential. Although naively this looks counterintuitive, it is important to realize that there are also situations, like LO and NNLO, where the regularity condition of the wave function conspires against the singularity so that the net effect is well behaved in the scattering amplitudes and deuteron properties. Our results in Sect. IV and V, suggest that the pattern obtained when comparing LO and NNLO looks quite converging numerically, although there appears to be no way of making an *a priori* estimate of the corrections.

Perturbative treatments might circumvent this difficulty since they have the indubitable benefit of allowing an *a priori* estimate of the systematic error via dimensional power counting. This causes no problem in the calculation of the long distance potential. However, we anticipate already that singular potentials are

indeed singular perturbations, and power counting may not work as one naively expects for the full amplitudes. Kaplan, Savage and Wise [68, 69] suggested such a perturbative scheme some years ago, where the lowest order approximation was a contact theory and OPE and higher order corrections could be computed in perturbation theory. This is equivalent to consider mM/f^2 to be first order and m^2/f^2 second order, so that a calculation involving the chiral constants would be N³LO in that counting. Unfortunately, the expansion turned out to be non-converging at NNLO [70]. In our coordinate space formulation, this approach corresponds to assume for the S waves a boundary condition fixing the scattering length α_0 [41] (See Appendix A of that work) and making long distance potential perturbations. In our previous work we verified that perturbation theory could only account for a contribution to the deuteron and $^3S_1 - ^3D_1$ scattering observables at first order. Unfortunately, the second order was divergent, while non-perturbatively, i.e., exactly solving the Schrödinger equation for the OPE potential, the results were not only finite but also numerically quite close to experiment. This deserves some explanation. In Fig. 15 we show the results in the deuteron channel when we scale the OPE potential $U_{\text{OPE}} \rightarrow \lambda U_{\text{OPE}}$ for the s-wave function normalization $A_S(\lambda)$ and the effective range $r_0(\lambda)$ as a function of the scaling parameter λ by keeping the deuteron binding energy fixed to its experimental value. The non-perturbative result is compared to the first order perturbation theory used in Ref. [41]. Clearly, perturbation theory fails even for weak coupling. The experimental value for r_0 could be obtained by adding a counterterm C_2 as done by Kaplan, Savage and Wise [68, 69]. A non-vanishing C_2 not only violates the orthogonality of the zero energy and deuteron wave functions for a long distance local potential but also introduces a new parameter, reducing the predictive power. Moreover, the non-perturbative inclusion of this C_2 counterterm with the OPE potential yields divergent results (see the discussion in Sect. VIIC). In fact, much of the strength of C_2 is naturally provided by the short distance $1/r^3$ singularity of the OPE potential.

B. Perturbations on the OPE potential

Recently, Nogga, Timmermans and van Kolck (NTvK) have suggested [42] treating the OPE effects non-perturbatively, i.e. to all orders, while TPE and higher as well as Δ contributions should be computed in perturbation theory (see also Refs. [55, 71, 72] for related ideas). In this section, we analyze such a proposal disregarding the Delta. Our main conclusions will not change although numbers could be modified. In such a situation the perturbative expansion is equivalent to consider the $mM/(4\pi f^2)$ to be zeroth order while $m^2/(4\pi f)^2$ is taken to be second order and m/M is first order, so that the

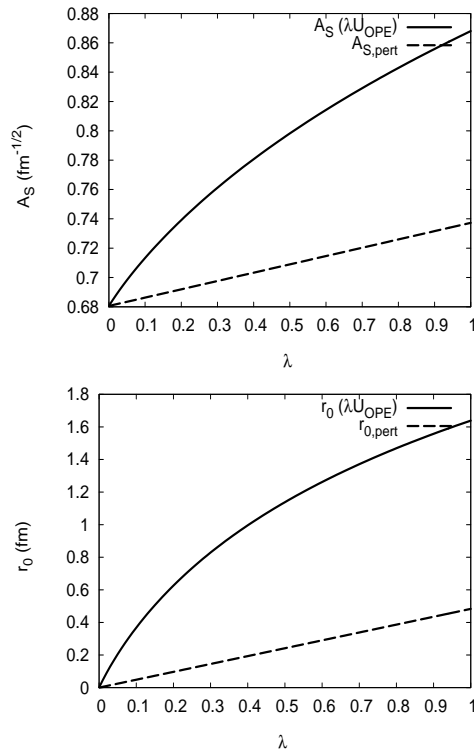


FIG. 15: Dependence of the s-wave function normalization $A_S(\lambda)$ in the deuteron and the effective range $r_0(\lambda)$ in the $^3S_1 - ^3D_1$ channel when one scales the OPE potential $U_{\text{LO}} \rightarrow \lambda U_{\text{LO}}$. In all cases we fix the deuteron binding energy to its experimental value.

potential can be written as

$$\begin{aligned} U^{(0)} &= U_{1\pi}^{(0)}, \\ U^{(2)} &= U_{1\pi}^{(2)} + U_{2\pi}^{(2)}, \\ U^{(3)} &= U_{1\pi}^{(3)} + U_{2\pi}^{(3)}. \end{aligned} \quad (96)$$

Non-perturbatively $U_{1\pi} = U_{1\pi}^{(0)} + U_{2\pi}^{(0)} + U_{3\pi}^{(0)} + \dots$ amounts to take $g_{\pi NN} = 13.1$ in the OPE piece, hence accounting for the Goldberger-Treiman discrepancy. In perturbation theory, we must take $U_{1\pi}^{(0)}$ with $g_A = 1.26$ and include OPE corrections to higher order. Note that the missing first order implies substantial simplifications in the perturbative treatment. Indeed NNLO can be done within first order perturbation theory since going to second order perturbation theory considers $U^{(2)}$ which is N³LO. In the remaining of this section we analyze some aspects of such proposal by scaling the strength of the perturbation and show that the appearance of non-analytical behaviour is intrinsic to singular potentials, yielding to perturbative divergences. As we will see, finite perturbative calculations including chiral TPE to NNLO would require 4 counterterms for the singlet 1S_0 and 6 counterterms for the triplet $^3S_1 - ^3D_1$ channel. Our non-perturbative results, i.e., fully iterated NNLO potentials,

are based on just 1 and 3 counterterms respectively.

C. Singlet 1S_0 channel in distorted OPE waves

Let us examine first the 1S_0 channel, and consider the effect of the NLO and NNLO TPE potentials on top of the LO OPE potential in long distance perturbation theory. The fact that they are taken second and third order respectively means that the effect will be additive at NNLO in the scattering properties. For the total potential in Eq. (96) we write the wave function as

$$u_k(r) = u_k^{(0)}(r) + u_k^{(2)}(r) + u_k^{(3)}(r) + \dots \quad (97)$$

and the phase shift becomes

$$\delta_0 = \delta_0^{(0)} + \delta_0^{(2)} + \delta_0^{(3)} + \dots \quad (98)$$

At LO the scattering length $\alpha_0^{(0)}$ is a free parameter which we fix to the physical value, $\alpha_0^{(0)} = \alpha_0$. As we did in our non-perturbative treatment in Sect. VIII A, we will keep the scattering length fixed to its experimental value at any order of the approximation, so that differences may be only attributable to the potential. In the normalization of Eq. (44) the correction to the phase shift is just given by

$$\delta_0^{(2)} = -k \sin^2 \delta_0^{(0)} \int_{r_c}^{\infty} U^{(2)}(r) u_k^{(0)}(r)^2 dr, \quad (99)$$

and a similar expression for $\delta_0^{(3)}$ which can be deduced by the standard Lagrange's identity. Here, a short distance cut-off r_c has been assumed because at short distances the NLO potential diverges as $U_{\text{NLO}} \sim 1/r^5$. The previous formula yields a change also in the scattering length, so that we may eliminate the cut-off radius by subtracting off the zero energy contribution by fixing $\alpha_0^{(2)} = 0$. It is convenient to recast the result in the form of an effective range expansion in the OPE distorted wave basis,

$$\begin{aligned} k \cot \delta_0 + \frac{1}{\alpha_0} &= k \cot \delta_0^{(0)} + \frac{1}{\alpha_0^{(0)}} \\ &+ \int_{r_c}^{\infty} dr U^{(2)}(r) \left[u_k^{(0)}(r)^2 - u_0^{(0)}(r)^2 \right] \\ &+ \int_{r_c}^{\infty} dr U^{(3)}(r) \left[u_k^{(0)}(r)^2 - u_0^{(0)}(r)^2 \right], \end{aligned} \quad (100)$$

which guarantees $\alpha_0^{(2)} = \alpha_0^{(3)} = 0$, due to the one subtraction. If we expand in powers of the energy the LO wave function we get

$$u_k^{(0)}(r) = u_0^{(0)}(r) + k^2 u_2^{(0)}(r) + k^4 u_4^{(0)}(r) + \dots \quad (101)$$

where

$$\begin{aligned} -u_0^{(0)''}(r) + U(r)u_0^{(0)}(r) &= 0 \\ -u_2^{(0)''}(r) + U(r)u_2^{(0)}(r) &= u_0^{(0)}(r) \\ -u_4^{(0)''}(r) + U(r)u_4^{(0)}(r) &= u_2^{(0)}(r) \\ &\dots \end{aligned} \quad (102)$$

These equations can be solved recursively. Thus, the NLO correction to the effective range is given by

$$r_0^{(2)} = 4 \int_{r_c}^{\infty} U^{(2)}(r) u_2^{(0)}(r) u_0^{(0)}(r) dr. \quad (103)$$

To estimate the short distance contribution we use the OPE exchange potential in the form $U_{\text{LO}} = -e^{-mr}/(R_s r)$, with $R_s = 16f^2\pi/g^2m^2M$ the characteristic length 1S_0 -channel scale. Note that the OPE potential in the 1S_0 channel is Coulomb like at short distances for which the complete regular plus irregular solution is known. One could then use a short distance expansion of the general analytical Coulomb solution. This facilitates guessing the solution at short distances for the zeroth energy wave function. The higher energy wave functions can be computed straightforwardly, yielding for $r \rightarrow 0$

$$\begin{aligned} u_0^{(0)}(r) &\sim c_0 \left[1 + mr - \frac{3r}{2R_s} - \frac{r}{R_s} \log \left(\frac{r}{R_s} \right) \right] + c_1 r \\ u_2^{(0)}(r) &\sim -c_0 r R_s + \mathcal{O}(r^3) \\ u_4^{(0)}(r) &\sim \frac{1}{3!} c_0 r^3 R_s + \mathcal{O}(r^5) \\ u_6^{(0)}(r) &\sim -\frac{1}{5!} c_0 r^5 R_s + \mathcal{O}(r^7) \\ u_8^{(0)}(r) &\sim \frac{1}{7!} c_0 r^7 R_s + \mathcal{O}(r^9) \end{aligned} \quad (104)$$

as can be readily checked by solving the Schrödinger equation in powers of energy, Eq. (102). The coefficients c_1 and c_0 correspond to the linearly independent regular and irregular solutions respectively and are determined by matching to the integrated in asymptotic condition $u_0^{(0)}(r) \rightarrow 1 - r/\alpha_0$ at large distances and at zero energy. Obviously, the irregular solution contributes, $c_0 \neq 0$, because α_0 is taken to be independent of the potential, and hence terms proportional to the coefficient c_1 are sub-leading. Thus, we get

$$r_0^{(2)} \sim 4 \int_{r_c}^{\infty} dr \frac{MC_5}{r^5} (-c_0^2 R_s) r. \quad (105)$$

Thus, we conclude that the first order perturbative result is badly divergent. This is very puzzling since the non-perturbative calculation in Sect. VIII A yields a finite number (see Eq. (89)), and suggests non-analytical dependence on the coupling constant. To enlighten the situation, let us scale the NLO potential by a factor λ , $U_{\text{NLO}} \rightarrow \lambda U_{\text{NLO}}$, and compute non-perturbatively the effective range as a function of the scaling parameters, $r_0(\lambda)$, with the obvious conditions $r_0(0) = r_0^{(0)}$ and

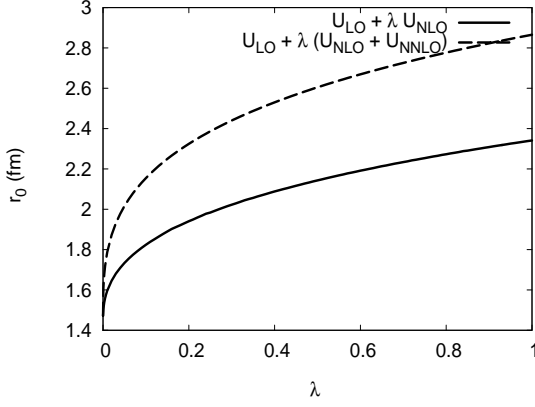


FIG. 16: Dependence of the effective range $r_0(\lambda)$ in the 1S_0 channel when the scaled potentials $U = U_{\text{LO}} + \lambda U_{\text{NLO}}$ and $U = U_{\text{LO}} + \lambda(U_{\text{NLO}} + U_{\text{NNLO}})$ are considered. In all cases we fix the scattering length to its experimental value.

$r_0(1) = r_0^{\text{NLO}}$. The result is presented in Fig. 16. The infinite slope at the origin can be clearly seen. Numerically, we find that for small $\lambda \ll 0.1$, the correction to the effective range behaves as $r_0 - r_0^{(0)} \sim \sqrt{\lambda}$, whereas for $\lambda \sim 1$, it behaves as $r_0 - r_0^{(0)} \sim \lambda^{1/3}$. This fractional power counting λ^α , with $0 < \alpha < 1$ is evident from the universal low energy theorem, Eq. (39), for a potential with a *single* scale, $U(r) = F(r/R)/R^2$, and appears when the short distance regulator is removed¹⁶. An explicit example is provided by Eq. (82) when a pure Van der Waals potential acts as a perturbation to a boundary condition (a contact theory with only α_0), since the strength of the potential is $\lambda MC_6 = R^4$, but r_0 contains $R \sim \lambda^{1/4}$, $R^2 \sim \lambda^{1/2}$ and $R^3 \sim \lambda^{3/4}$. It would be interesting to predict *a priori* this non-perturbative non-integer power counting analytically for potentials with multiple scales as we have done here numerically [67]¹⁷

Obviously, to prevent the perturbative divergence one could subtract an energy dependent contribution, and provide the effective range as an input parameter¹⁸.

Then one would get

$$k \cot \delta = k \cot \delta^{(0)} + \frac{1}{2} (r_0 - r_0^{(0)}) k^2 + \int_{r_c}^{\infty} dr [U^{(2)}(r) + U^{(3)}(r)] \times [u_k^{(0)}(r)^2 - u_0^{(0)}(r)^2 - 2k^2 u_0^{(0)}(r) u_2^{(0)}(r)] / 6$$

Note that this equation requires assuming $r_0 - r_0^{(0)} = \mathcal{O}(\lambda)$, while non-perturbatively we find $r_0 - r_0^{(0)} = \mathcal{O}(\lambda^{1/2})$. Now the NLO and NNLO corrections to the v_2 parameter would come as a prediction,

$$v_2^{(2)} + v_2^{(3)} = \int_{r_c}^{\infty} dr [U^{(2)}(r) + U^{(3)}(r)] \times [2u_4^{(0)}(r)u_0^{(0)}(r) + u_2^{(0)}(r)^2], \quad (107)$$

which is also divergent since the leading behaviour of the integrand is $\sim 1/r^3$ at NLO and $\sim 1/r^4$ at NNLO for small r , see Eq. (104). Thus, a further subtraction would be needed, predicting the correction to v_3 ,

$$v_3^{(2)} + v_3^{(3)} = \int_{r_c}^{\infty} dr [U^{(2)}(r) + U^{(3)}(r)] \times [2u_6^{(0)}(r)u_0^{(0)}(r) + 2u_2^{(0)}(r)u_4^{(0)}(r)], \quad (108)$$

which is logarithmically divergent due to Eq. (104). Finally, if a fourth subtraction is implemented a convergent prediction is obtained for v_4 at NLO and NNLO,

$$v_4^{(2)} + v_4^{(3)} = \int_{r_c}^{\infty} dr [U^{(2)}(r) + U^{(3)}(r)] \times [2u_8^{(0)}(r)u_0^{(0)}(r) + 2u_2^{(0)}(r)u_4^{(0)}(r) + u_4^{(0)}(r)^2]. \quad (109)$$

These four subtractions, needed to make a renormalized *perturbative* prediction of the 1S_0 phase shift at NNLO, actually correspond to having 4 counterterms, i.e. fixing α_0, r_0, v_2 and v_3 . This result disagrees with the standard Weinberg counting (two counterterms at NLO and NNLO in the 1S_0 channel). Moreover, besides the loss of predictive power as compared to the non-perturbative result where only one counterterm is needed the deduced renormalized value for v_4 is worsened in perturbation theory, since $v_4^{(0)} = -50.74 \text{ fm}^7$, $v_4^{(2)} = -10.45 \text{ fm}^7$ and $v_4^{(3)} = -2.88 \text{ fm}^7$. The situation is summarized in Table VII where we show our numerical results obtained in perturbation theory as explained above and they are compared to the NijmII and Reid93 potential model calculations (See e.g. Ref. [57]). Although these are not directly experimental data it is noteworthy that they differ by a few percent while the perturbative calculation is about a factor 3 larger. The integrals for v_4 are rather

¹⁶ In Ref. [48] a potential well was used as a short distance regulator which was not removed, and hence the non-analyticity was not seen.

¹⁷ Fractional power counting has also been reported to occur also in the EFT analysis of the three body problem for the pionless theory [55, 72].

¹⁸ This is equivalent to use a short distance energy dependent boundary condition on the solution and hence to violate the orthogonality conditions discussed in Sect. III.

TABLE VII: Threshold parameters of the effective range expansion $k \cot \delta = -1/\alpha_0 + r_0 k^2/2 + v_2 k^4 + v_3 k^6 + v_4 k^8$ in the singlet 1S_0 channel in OPE distorted waves perturbation theory. We take $U^{(0)} = U_{1\pi}^{(0)}$ with $g_A = 1.26$. For the NNLO case we use Set IV for the chiral constants c_1 , c_3 and c_4 given in Table I.

1S_0	LO	NLO _{pert}	N ² LO _{pert}	Exp.	Nijm II	Reid93
α_0 (fm)	Input	Input	Input	-23.74(2)	-23.73	-23.74
r_0 (fm)	1.383	Input	Input	2.77(5)	2.67	2.75
v_2 (fm ³)	-2.053	Input	Input	—	-0.48	-0.49
v_3 (fm ⁵)	9.484	Input	Input	—	3.96	3.65
v_4 (fm ⁷)	-50.74	-61.19	-64.07	—	-19.88	-18.30

well converging and the matching between the numerical solution and the short distance solutions, Eq. (104), is quite stable in the region around $r \sim 0.1$ fm.

Thus, in this particular example of the 1S_0 channel one sees that our non-perturbative approach based on the choice of the regular solutions at the origin predicts the phase shift and hence all low energy parameters from α_0 and the potential as displayed in Table VI. A perturbative treatment of the amplitude based on OPE distorted waves requires to fix α_0, r_0, v_2 and v_3 at NNLO. The phenomenological success and converging pattern observed when the potential is considered at LO, NLO and NNLO is solved non-perturbatively is very encouraging. The price to pay is to face non-analytical behaviour which implies a non-integer power counting. The trend observed here can be generalized to other channels. A more thorough discussion of this issue will be presented elsewhere [67].

D. Triplet $^3S_1 - ^3D_1$ channel in distorted OPE waves

We turn now to the triplet $^3S_1 - ^3D_1$ channel. The reasoning is a straightforward, although tedious, coupled channel generalization of the 1S_0 case, with the additional feature that the short distance behaviour is dominated by a $1/r^3$ singularity (instead of $1/r$), and so the short distance behaviour is different. It is convenient to introduce the potential matrix as

$$\mathbf{U}(r) = \begin{pmatrix} U_{3S_1}(r) & U_{E_1}(r) \\ U_{E_1}(r) & U_{3D_1}(r) \end{pmatrix}, \quad (110)$$

and the matrix wave function,

$$\mathbf{u}_k(r) = \mathbf{A} \begin{pmatrix} u_{k,\alpha}(r) & u_{k,\beta}(r) \\ w_{k,\alpha}(r) & w_{k,\beta}(r) \end{pmatrix}, \quad (111)$$

with \mathbf{A} a constant energy dependent matrix, subject to a slightly different normalization than Eq. (75),

$$\mathbf{u}_k(r) \rightarrow \frac{1}{k} \hat{\mathbf{j}}(kr) \mathbf{D}^{-1} \hat{\mathbf{M}} - \hat{\mathbf{y}}(kr) \mathbf{D} \quad (112)$$

Here, $\hat{\mathbf{M}}$ is the effective range matrix defined by its relation to the unitary \mathbf{S} -matrix,

$$\mathbf{DSD}^{-1} = \left(\hat{\mathbf{M}} + ik\mathbf{D}^2 \right) \left(\hat{\mathbf{M}} - ik\mathbf{D}^2 \right)^{-1}, \quad (113)$$

and $\mathbf{D} = \text{diag}(1, k^2)$. The reduced Bessel functions matrices are given by $\hat{\mathbf{j}} = \text{diag}(\hat{j}_0, \hat{j}_2)$ and $\hat{\mathbf{y}} = \text{diag}(\hat{y}_0, \hat{y}_2)$ with $\hat{j}_l(x) = x j_l(x)$ and $\hat{y}_l(x) = x y_l(x)$. At low energies, one has the effective range expansion (see e.g. [57] and references therein),

$$\hat{\mathbf{M}} = -(\mathbf{a})^{-1} + \frac{1}{2} \mathbf{r} k^2 + \mathbf{v} k^4 + \dots \quad (114)$$

Here, we have introduced the scattering length matrix,

$$\mathbf{a} = \begin{pmatrix} \alpha_0 & \alpha_{02} \\ \alpha_{02} & \alpha_2 \end{pmatrix}, \quad (115)$$

the effective range matrix

$$\mathbf{r} = \begin{pmatrix} r_0 & r_{02} \\ r_{02} & r_2 \end{pmatrix}, \quad (116)$$

and so on. These parameters have been determined in [57] from the potentials of Ref. [44]. Proceeding similarly as in the one channel case, one gets, after one subtraction at zero energy the effective range function in perturbation theory,

$$\begin{aligned} \hat{\mathbf{M}} + (\mathbf{a})^{-1} &= \hat{\mathbf{M}}^{(0)} + (\mathbf{a}^{(0)})^{-1} \\ &+ \int_{r_c}^{\infty} dr \left[\mathbf{u}_k^{(0)\dagger} \mathbf{U}^{(2)} \mathbf{u}_k^{(0)} - \mathbf{u}_0^{(0)\dagger} \mathbf{U}^{(2)} \mathbf{u}_0^{(0)} \right] \end{aligned} \quad (117)$$

The condition $\alpha_0^{(0)} = \alpha_0$ must be imposed, since $\alpha_{02}^{(0)}$ and $\alpha_2^{(0)}$ are predicted from $\alpha_0^{(0)}$ (at LO one only needs one counterterm). This formula implies that one introduces two new conditions to fix now α_{02} and α_2 to their experimental value. Along similar lines as done before, we analyze the finiteness of the previous expression by computing the effective range matrix. To this end we expand the coupled channel wave function in powers of momentum

$$\mathbf{u}_k^{(0)}(r) = \mathbf{u}_0^{(0)}(r) + k^2 \mathbf{u}_2^{(0)}(r) + k^4 \mathbf{u}_4^{(0)}(r) + \dots \quad (118)$$

to get

$$\mathbf{r}^{(2)} = \int_{r_c}^{\infty} dr \left[\mathbf{u}_2^{(0)\dagger} \mathbf{U}^{(2)} \mathbf{u}_0^{(0)} + \mathbf{u}_0^{(0)\dagger} \mathbf{U}^{(2)} \mathbf{u}_2^{(0)} \right] \quad (119)$$

The LO OPE short distance behaviour of the triplet wave functions has been worked out in our previous work [41]. It is convenient to define the triplet length scale

$$R_t = \frac{3g_A^2 M}{32\pi f_\pi^2} \quad (120)$$

which value $R_t = 1.07764\text{fm}$. One has the general structure

$$\begin{aligned}
 u(r) &= \frac{1}{\sqrt{3}} \left(\frac{r}{R_t} \right)^{3/4} \left[-C_{1R} f_{1R}(r) e^{+4\sqrt{2}\sqrt{\frac{R_t}{r}}} \right. \\
 &\quad - C_{2R} f_{2R}(r) e^{-4\sqrt{2}\sqrt{\frac{R_t}{r}}} + \sqrt{2} C_{1A} f_{1A}(r) e^{-4i\sqrt{\frac{R_t}{r}}} \\
 &\quad \left. + \sqrt{2} C_{2A} f_{2A}(r) e^{4i\sqrt{\frac{R_t}{r}}} \right] \\
 w(r) &= \frac{1}{\sqrt{3}} \left(\frac{r}{R_t} \right)^{3/4} \left[\sqrt{2} C_{1R} g_{1R}(r) e^{+4\sqrt{2}\sqrt{\frac{R_t}{r}}} \right. \\
 &\quad + \sqrt{2} C_{2R} g_{2R}(r) e^{-4\sqrt{2}\sqrt{\frac{R_t}{r}}} + C_{1A} g_{1A}(r) e^{-4i\sqrt{\frac{R_t}{r}}} \\
 &\quad \left. + C_{2A} g_{2A}(r) e^{4i\sqrt{\frac{R_t}{r}}} \right]
 \end{aligned} \tag{121}$$

where the constants C_{1R} , C_{2R} , C_{1A} and C_{2A} depend on the energy and the OPE potential parameters. The regular solution is selected when one takes $C_{1R} = 0$. The functions appearing in this formula are of the form

$$\begin{aligned}
 f(r) &= \sum_{n=0}^{\infty} a_n \left(\frac{r}{R_t} \right)^{n/2} \\
 g(r) &= \sum_{n=0}^{\infty} b_n \left(\frac{r}{R_t} \right)^{n/2}
 \end{aligned} \tag{122}$$

For the present calculation we only need the power behaviour (see Appendix B of Ref. [41])

$$\begin{aligned}
 \mathbf{u}_0^{(0)}(r) &\sim r^{3/4} \\
 \mathbf{u}_2^{(0)}(r) &\sim r^{3/4+5/2} \\
 \mathbf{u}_4^{(0)}(r) &\sim r^{3/4+5}
 \end{aligned} \tag{123}$$

which shows that, again, the first order correction to the effective range matrix is logarithmically divergent because the NLO potential diverges as $1/r^5$ and $\mathbf{U}^{(2)}\mathbf{u}_0\mathbf{u}_2 \sim 1/r$ ¹⁹. As previously, the situation could be amended by adding 3 new counterterms to fix the effective range matrix \mathbf{r} and then \mathbf{v} would come as a prediction. So, at NLO in perturbation theory one needs a total of 6 counterterms to generate a coupled channel finite amplitude. When adding the NNLO contribution this number of counterterms remains the same since $\mathbf{U}^{(3)}\mathbf{u}_0^{(0)}\mathbf{u}_4^{(0)} \sim r^{1/2}$ and $\mathbf{U}^{(3)}[\mathbf{u}_2^{(0)}]^2 \sim r^{1/2}$.

An illustration of non-analytical non-perturbative behaviour in the ${}^3S_1 - {}^3D_1$ -channel can be looked up in

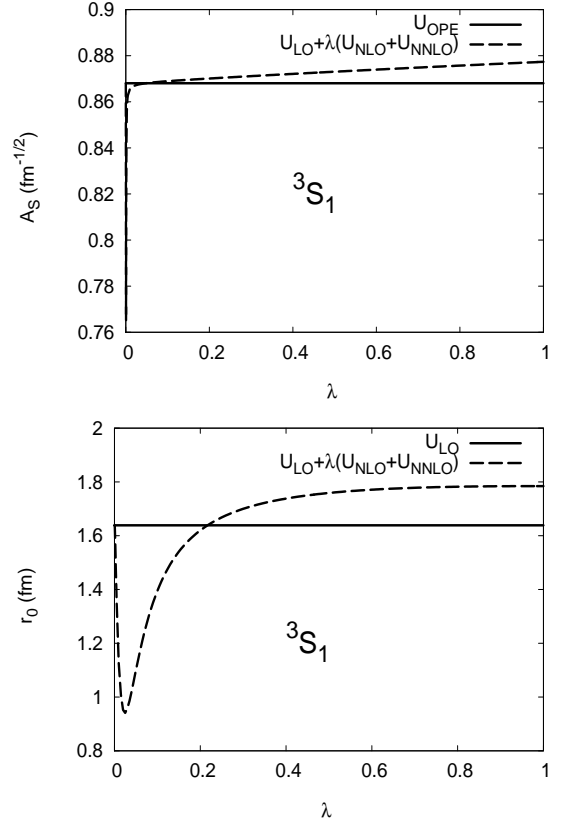


FIG. 17: Dependence of the s-wave function normalization $A_S(\lambda)$ in the deuteron and the effective range $r_0(\lambda)$ in the ${}^3S_1 - {}^3D_1$ channel when one scales the TPE potential $U = U_{LO} + \lambda(U_{NLO} + U_{NNLO})$. In all cases we fix the deuteron binding energy, the asymptotic D/S -ratio, η and the s-wave scattering length, α_0 are fixed to their experimental values.

Fig. 17. There, the behaviour of the s-wave function normalization $A_S(\lambda)$ in the deuteron and the effective range $r_0(\lambda)$ in the ${}^3S_1 - {}^3D_1$ channel when the TPE potential is scaled as $U = U_{LO} + \lambda(U_{NLO} + U_{NNLO})$ and the deuteron binding energy, the asymptotic D/S -ratio, η and the s-wave scattering length, α_0 are fixed to their experimental values.

E. The deuteron in distorted OPE waves

To conclude our analysis of perturbation theory we study now the deuteron bound state. According to Fig. 17 there appears some tiny non-analyticity for very small couplings in the asymptotic s-wave normalization A_S . Note that there is an apparent linear behaviour with the exception of the very small λ region, making one suspect that the result might be obtained in perturbation theory. We will see below by an explicit perturbative calculation that *this is not so*. We have checked that this trend also occurs for other quantities such as the

¹⁹ There is a subtlety here. The terms containing the regular exponential at the origin are convergent, regardless on the power of r in the denominator. Naively logarithmically divergent integrals would become convergent when combined with oscillating functions. However, these functions appear *squared* so that the logarithmic divergence prevails.

quadrupole moment, Q_d , the matter radius, r_m , and the D-state probability, P_D . Here, we show, as it has been done above for the scattering problem, that this can be traced to a first order divergent renormalized result.

We define the two component deuteron state,

$$\mathbf{u}_\gamma(r) = \begin{pmatrix} u_\gamma(r) \\ w_\gamma(r) \end{pmatrix} \quad (124)$$

In perturbation theory, we expand the potential

$$\mathbf{U}(r) = \mathbf{U}^{(0)}(r) + \mathbf{U}^{(2)}(r) + \mathbf{U}^{(3)}(r) + \dots \quad (125)$$

and thus the deuteron wave function for fixed energy (or γ),

$$\mathbf{u}_\gamma(r) = \mathbf{u}_\gamma^{(0)}(r) + \mathbf{u}_\gamma^{(2)}(r) + \mathbf{u}_\gamma^{(3)}(r) + \dots \quad (126)$$

where $(u_\gamma^{(0)}(r), w_\gamma^{(0)}(r))$ correspond to the lowest order solutions of the problem and $(u_\gamma^{(2)}(r), w_\gamma^{(2)}(r))$ and $(u_\gamma^{(3)}(r), w_\gamma^{(3)}(r))$ satisfy

$$\begin{aligned} -\mathbf{u}_\gamma^{(0)}(r) + [\mathbf{U}^{(0)}(r) + \gamma^2] \mathbf{u}_\gamma^{(0)}(r) &= 0 \\ -\mathbf{u}_\gamma^{(2)}(r) + [\mathbf{U}^{(0)}(r) + \gamma^2] \mathbf{u}_\gamma^{(2)}(r) &= -\mathbf{U}^{(2)}(r) \mathbf{u}_\gamma^{(0)}(r) \\ -\mathbf{u}_\gamma^{(3)}(r) + [\mathbf{U}^{(0)}(r) + \gamma^2] \mathbf{u}_\gamma^{(3)}(r) &= -\mathbf{U}^{(3)}(r) \mathbf{u}_\gamma^{(0)}(r) \end{aligned} \quad (127)$$

We look for normalized solutions, so that perturbatively

$$\begin{aligned} 1 &= \int_0^\infty dr \mathbf{u}_\gamma^{(0)\dagger}(r) \mathbf{u}_\gamma^{(0)}(r) \\ 0 &= \int_0^\infty dr \left(\mathbf{u}_\gamma^{(2)\dagger}(r) \mathbf{u}_\gamma^{(0)}(r) + \mathbf{u}_\gamma^{(0)\dagger}(r) \mathbf{u}_\gamma^{(2)}(r) \right) \\ 0 &= \int_0^\infty dr \left(\mathbf{u}_\gamma^{(3)\dagger}(r) \mathbf{u}_\gamma^{(0)}(r) + \mathbf{u}_\gamma^{(0)\dagger}(r) \mathbf{u}_\gamma^{(3)}(r) \right) \end{aligned} \quad (128)$$

The zeroth order equation was solved in our previous work [41], where it was shown that γ was a free parameter, which means $\gamma^{(0)} = \gamma$, and the regular solution at the origin was selected (see Eq. (121)) to ensure normalizability at the origin. We will keep always the same fixed value at any order of the approximation, so that $\gamma^{(2)} = \gamma^{(3)} = 0$. To analyze the NLO and NNLO problem analytically we proceed by the variable coefficients method. The zeroth order equation is a homogenous linear system with four linearly independent solutions,

$$\mathbf{u}_i^{(0)}(r) = \begin{pmatrix} u_i(r) \\ w_i(r) \end{pmatrix} \quad i = 1, 2, 3, 4 \quad (129)$$

The first order equation is an inhomogeneous linear system, which solution can be written as

$$\begin{aligned} u_\gamma^{(2)}(r) + u_\gamma^{(3)}(r) &= \sum_{i=1}^4 c_i(r) u_i(r) \\ w_\gamma^{(2)}(r) + w_\gamma^{(3)}(r) &= \sum_{i=1}^4 c_i(r) w_i(r) \end{aligned} \quad (130)$$

The variable coefficients satisfy

$$\begin{aligned} \sum_{i=1}^4 c_i'(r) u_i(r) &= 0 \\ \sum_{i=1}^4 c_i'(r) w_i(r) &= 0 \\ \sum_{i=1}^4 c_i'(r) u_i'(r) &= F_u(r) \\ \sum_{i=1}^4 c_i'(r) w_i'(r) &= F_w(r) \end{aligned} \quad (131)$$

where we have defined the driving term

$$\begin{aligned} \mathbf{F}(r) &= \begin{pmatrix} F_u(r) \\ F_w(r) \end{pmatrix} \\ &= -\mathbf{U}^{(2)}(r) \mathbf{u}_\gamma^{(0)}(r) - \mathbf{U}^{(3)}(r) \mathbf{u}_\gamma^{(0)}(r) \end{aligned} \quad (132)$$

which at short distances behaves as

$$\mathbf{F}(r) \sim r^{3/4} C_5 r^{-5} + r^{3/4} C_6 r^{-6} \quad (133)$$

whereas at large distances one has

$$\mathbf{F}(r) \sim e^{-\gamma r} e^{-mr} \quad (134)$$

To proceed further, we choose the following linearly independent solutions fulfilling the asymptotic boundary condition at infinity

$$\begin{aligned} u_1(r) &\rightarrow e^{-\gamma r}, \\ w_1(r) &\rightarrow 0, \\ u_2(r) &\rightarrow 0, \\ w_2(r) &\rightarrow e^{-\gamma r} \left(1 + \frac{3}{\gamma r} + \frac{3}{(\gamma r)^2} \right), \\ u_3(r) &\rightarrow e^{\gamma r}, \\ w_3(r) &\rightarrow 0, \\ u_4(r) &\rightarrow 0, \\ w_4(r) &\rightarrow e^{\gamma r} \left(1 - \frac{3}{\gamma r} + \frac{3}{(\gamma r)^2} \right), \end{aligned} \quad (135)$$

Any of these solutions has a short distance behaviour of the general form given in Eq. (121). So that, *all these solutions are necessarily singular* at the origin. Using Krammer's rule the solutions to the linear differential system, Eq. (131), which are regular at infinity read

$$c_1(r) = \frac{1}{W} \int_0^r dr' \begin{vmatrix} 0 & u_2 & u_3 & u_4 \\ 0 & w_2 & w_3 & w_4 \\ F_u & u_2' & u_3' & u_4' \\ F_w & w_2' & w_3' & w_4' \end{vmatrix} \quad (136)$$

$$c_2(r) = \frac{1}{W} \int_0^r dr' \begin{vmatrix} u_1 & 0 & u_3 & u_4 \\ w_1 & 0 & w_3 & w_4 \\ u'_1 & F_u & u'_3 & u'_4 \\ w'_1 & F_w & w'_3 & w'_4 \end{vmatrix} \quad (137)$$

$$c_3(r) = -\frac{1}{W} \int_r^\infty dr' \begin{vmatrix} u_1 & u_2 & 0 & u_4 \\ w_1 & w_2 & 0 & w_4 \\ u'_1 & u'_2 & F_u & u'_4 \\ w'_1 & w'_2 & F_w & w'_4 \end{vmatrix} \quad (138)$$

$$c_4(r) = -\frac{1}{W} \int_r^\infty dr' \begin{vmatrix} u_1 & u_2 & u_3 & 0 \\ w_1 & w_2 & w_3 & 0 \\ u'_1 & u'_2 & u'_3 & F_u \\ w'_1 & w'_2 & w'_3 & F_w \end{vmatrix} \quad (139)$$

where W is the Wronskian

$$W = \begin{vmatrix} u_1 & u_2 & u_3 & u_4 \\ w_1 & w_2 & w_3 & w_4 \\ u'_1 & u'_2 & u'_3 & u'_4 \\ w'_1 & w'_2 & w'_3 & w'_4 \end{vmatrix} = -4\gamma^2. \quad (140)$$

At asymptotically large distances we have

$$\begin{aligned} u^{(2)}(r) &\rightarrow c_S^{(2)} e^{-\gamma r} \\ w^{(2)}(r) &\rightarrow c_D^{(2)} \eta^{(0)} e^{-\gamma r} \left(1 + \frac{3}{\gamma r} + \frac{3}{(\gamma r)^2} \right) \end{aligned} \quad (141)$$

and similarly for the N²LO correction. Note that the normalization condition, Eq. (128), implies a linear relation between $c_S^{(2)}$ and $c_D^{(2)}$ as well as $c_S^{(3)}$ and $c_D^{(3)}$. The total D/S ratio obtained by including the zeroth order contribution is given by

$$\eta = \eta^{(0)} \frac{1 + c_D^{(2)} + c_D^{(3)}}{1 + c_S^{(2)} + c_S^{(3)}} \quad (142)$$

If we fix η we get a relation between c_S and c_D . The coefficients $c_S^{(2)}$ and $c_D^{(2)}$ are given by

$$\begin{aligned} c_S^{(2)} + c_S^{(3)} &= c_1(\infty) \\ \eta^{(0)} (c_D^{(2)} + c_D^{(3)}) &= c_2(\infty) \end{aligned} \quad (143)$$

The long distance behaviour of the integrands is well behaved since, up to inessential powers in r , one has

$$\begin{aligned} c'_1(r) &\sim e^{-2mr} \\ c'_2(r) &\sim e^{-2mr} \\ c'_3(r) &\sim e^{-(2m+2\gamma)r} \\ c'_4(r) &\sim e^{-(2m+2\gamma)r} \end{aligned} \quad (144)$$

However, the leading short distance behaviour of the integrand is given

$$c'_i(r) \sim r^{3/4} (e^{4\sqrt{2R/r}})^2 \left(\frac{C_5}{r^5} + \frac{C_6}{r^6} \right) r^{3/4} e^{\pm i4\sqrt{R/r}} \quad (145)$$

So, we expect the coefficients c_S and c_D to diverge if the short distance cut-off is removed, $r_c \rightarrow 0$. It is unclear how this divergence might be avoided. Unlike the scattering problem in perturbation theory, where energy dependent (and hence orthogonality violating) subtractions are needed, it would be difficult to accept a bound state not normalized to unity unless one includes, besides pn other Fock state components, such as $pn\pi$. This short distance analysis holds also when the $1/r^6$ Δ -contributions are taken into account perturbatively.

Thus, perturbation theory on the distorted OPE basis for the deuteron makes sense only as a finite cut-off theory. In the appendix A we develop further such an approach to NLO, where η is an input and to NNLO, where *both* A_S and η should be fixed. We also show that in the cut-off theory the NLO yields tiny corrections to deuteron properties whereas the NNLO dominates. This proves that, at least perturbatively and in the absence of the Δ , the (integer) power counting to NNLO in deuteron properties is obviously not convergent. To some extent this result resembles qualitatively the findings of Ref. [70] based on the idea that OPE and TPE can be included perturbatively [68]. Of course, an amelioration of the convergence in the cut-off theory when Δ 's are included is not precluded, and deserves further investigation. However, the very need of a finite cut-off will still hold, as our analytical study shows.

In Fig. 18 we show LO, NLO and NNLO order wave functions when a finite short distance cut-off $r_c = 0.5\text{fm}$ is considered. The strong divergence of the wave function at the origin can be clearly seen.

X. CONCLUSIONS

In the present work we have extended the coordinate space renormalization of central waves in NN interaction discussed in our previous work [41] for the OPE potential to the TPE potential. As we have stressed along the paper, the main advantage of such a framework is that the (renormalized) potential is finite everywhere except at the origin where a Van der Waals attractive singularity takes place. This suggests using a radial cut-off which provides a compact support for the short range part of the potential thus making scheme dependent contact interactions innocuous for the long range solution. Thus, there is no need to devise different regularization methods for the potential and the wave functions both for bound state and scattering state solutions. As a result model independent long range correlations between NN observables can be deduced if the potential is iterated to all orders.

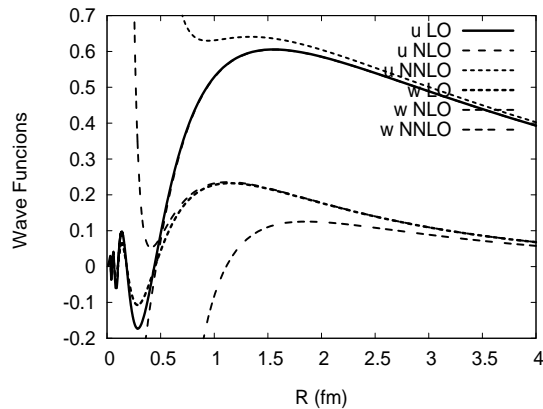


FIG. 18: Deuteron wave functions at LO, NLO and NNLO in perturbation theory when a short distance cut-off $r_c = 0.5\text{fm}$ is considered for the perturbative corrections. At LO we fix γ , at NLO we fix γ and η and at NNLO γ , η and A_S are fixed. Solutions are perturbatively normalized. The LO wave functions are normalized by taking $A_S = 1$

Important constraints can be deduced from the requirement of a small wave function in the unknown short distance region. As a consequence the boundary condition for the wave function at short distances becomes energy independent if the long range contribution to the potential is also energy independent. We stress here that such requirements, although quite natural from a physical viewpoint, may not appear obvious within the so far established EFT framework, and it would be very interesting to provide further arguments within EFT itself supporting our unconventional framework [56]. Actually, we find that the singularity structure of the potential at short distances determines uniquely how many parameters must be regarded as unknown, non predictable, information. This is done in terms of short distance phases or equivalently via suitable mixed boundary conditions at the origin. Moreover, for an energy independent potential the orthogonality of wave functions precludes a possible energy dependence of the boundary conditions. In the particular cases studied in this paper, namely 1S_0 and $^3S_1 - ^3D_1$ channels, we have found that besides the NNLO TPE potential parameters, one can use the S-wave scattering lengths in both channels as well as the deuteron binding energy and the asymptotic D/S ratio of the deuteron wave functions as independent input information. The remaining scattering or bound state properties in the triplet channel are then predicted unambiguously. Based on the superposition principle of boundary conditions, we have found analytical and simple universal rational relations which clearly exhibit these features. These universal relations would be very difficult to deduce in momentum space and, moreover, are free from uncertainties attributable to finite cut-off effects. So, the cut-off has been effectively eliminated. On a numerical level, the fact that our problem is an initial value problem

for the Schrödinger equation starting at infinity, makes possible to obtain any solution by competitive algorithms with adaptable integration steps with any prescribed accuracy. This allows to faithfully describe the short distance oscillations of the wave function. This is in contrast to the standard Lippmann-Schwinger treatments, where matrix inversion methods may eventually run into computer space limitations with a natural loss of space resolution as a side-effect. The non-trivial oscillating structure of the wave functions with ever decreasing periods of the wave functions close to the origin would actually be very difficult to reproduce within a momentum space framework.

According to our analysis, there are finite cut-off effects in previous works dealing also with TPE potentials both in coordinate as well as in momentum space. The induced corrections are larger than the experimental uncertainty of the computed observables, so that in some cases agreement with data may be clearly attributed to the choice of a finite cut-off. In our energy independent boundary condition treatment we found short distance cut-offs of about $r_c = 0.1 - 0.2\text{fm}$ to be rather innocuous. Within a Wilsonian viewpoint of renormalization, changes in the cut-off should correspond to decimation, i.e. halving, and not to linear changes in the scale. If one associates this coordinate space cut-off to a momentum space ultraviolet cut-off of $\Lambda = \pi/2r_c$ [73] we are dealing with an equivalent momentum scale of about $1.5 - 3\text{GeV}$, much larger than the scales below 1GeV usually employed in momentum space calculations where only linear sensitivity to changes of the cut-off was implemented. Nevertheless, it is fair to say that the calculations based on Sets III and IV provide not too large discrepancies.

As one naturally expects in a renormalized theory, errors are dominated by uncertainties in the input data, and not by cut-off uncertainties. Indeed, we seem to reach a limit in the accuracy of the predictions, paralleling the findings in ChPT for mesons at the two loop level. At the OPE level, one can predict bound state and scattering properties in the singlet 1S_0 and triplet $^3S_1 - ^3D_1$ channels solely from the deuteron energy and the 1S_0 scattering length. At the TPE level, one needs not only the additional chiral constants c_1 , c_3 and c_4 but also the triplet S-wave scattering length and the asymptotic D/S deuteron ratio. Although the TPE central values predictions improve, the induced TPE errors turn out to be *larger* than the OPE uncertainties. In fact, this large uncertainties make that, within errors, the TPE calculation becomes compatible with experimental data at the 1σ level. This suggests that in order to see in a statistically significant sense other effects, such as electromagnetic, relativistic and three pion effects one must first improve on the input data. Otherwise, predictive power is lost. Nevertheless, given the finite cut-off effects detected in previous works, the role of these corrections beyond TPE should be reanalyzed within the present approach.

One of the important consequences of our treatment is that the chiral constants c_1 , c_3 and c_4 can be determined

from *low energy data* and *deuteron properties*. Specifically, we have used the singlet and triplet effective ranges as well as the asymptotic S-wave deuteron wave function to c_1 , c_3 and c_4 with errors varying all input data within their experimental uncertainties. The decision on what set of data should be used to pin down the chiral coefficients is not entirely trivial, because it should become clear which hypothesis we want to verify or to refute. The absence of cut-off effects makes this test cleaner; we just check whether the TPE potential holds from zero to infinity. Obviously, this cannot be literally true, but one expects that at low energies other short range effects can be considered negligible. Let us remind that error analysis within NN calculations was only carried out in a large scale partial wave analysis to data in Ref. [11]. The determinations of chiral constants based on a fit to NN databases [43, 44, 45] for phase shifts lack any error estimates because the databases themselves are treated as errorless. The determination of chiral constants from peripheral waves has similar drawbacks. From the chiral theory point of view we see that it is possible to determine these parameters precisely in the regime where we trust the theory most, namely in the description of low energy NN data. A fit becomes possible, and the values it yields only differ by 2σ with the determination from πN data. We do not exclude that our values for the chiral constants may eventually spoil the successful overall fit of phase shifts in all channels presented in the past, after all renormalization has been carried out. If so, the situation on the effectiveness of effective field theory would be in a less optimistic shape than assumed hitherto. A preliminary analysis of the problem shows what Van der Waals coefficients in the TPE potential correspond to attractive short range interactions, and hence what phase shifts are completely determined in terms of coupled channel scattering lengths. This issue is very relevant and would require a detailed channel by channel analysis and renormalization, taking as input the scattering lengths documented in our previous work [57] and integrating in from large distances along the lines of the present approach. Full details are reported elsewhere [74].

Nevertheless, despite of the good convergence in the 1S_0 channel for LO, NLO and NNLO calculations, we have noted a difficulty for the triplet $^3S_1 - ^3D_1$ channel at NLO of the potential. In contrast to dimensional power counting expectations one cannot use the scattering length α_0 , the effective range r_0 and α_{02} as arbitrary input parameters at NLO in the potential (one could equally take γ , η and α_0) but they are entirely predicted from the potential as required by finiteness of the phase-shifts. Otherwise, the scattering amplitude diverges, as we have shown. We have also seen that even if one assumes a finite value of the cut-off the NLO is worse than the LO, suggesting that the problem may indeed be related to the power counting on the long distance potential. Remarkably, these parameters must be fixed at NNLO where, according to the standard approach, no further low energy parameters should be fixed. This mis-

match in orders can be understood if one considers the $N\Delta$ splitting to be a small parameter, making much of the NNLO contributions to the potential become NLO ones, because c_3 and c_4 would be order minus one. In such a case, our interpretation goes hand in hand with the standard approach; one needs three independent low energy parameters at NLO in this counting. The consequences of this Δ -counting to higher orders within the context of renormalization will be explored elsewhere. Of course, we should point out that despite the rather tantalizing description achieved at NNLO the existence of a consistent power counting guaranteeing the success of the present approach to all orders remains to be proved. A key ingredient of such a power counting would be the correct incorporation of all long range physics. Apparently, within Weinberg's power counting the NLO in the deuteron channel misses important contributions.

Finally, we have analyzed the consequences of a perturbative expansion of TPE effects taking the OPE results as a zeroth order approximation as suggested recently [42]. Our non-perturbative calculations based on iterating a perturbative potential to all orders exhibit unequivocal non-analytic dependence on the expansion parameter, due to the singular character of the chiral potentials at the origin. This is equivalent to a non-integer enhancement of the power counting λ^α with $0 < \alpha < 1$ in the potential, and it would be interesting to know the general rules of such a counting *a priori* [67]. Thus, perturbation theory based on standard power counting becomes divergent and can only yield finite results at the expense of introducing more perturbative counterterms than are needed in a non-perturbative treatment. This is just a manifestation of the fact that singular potentials require infinite counterterms in perturbation theory, while only a few ones are needed non-perturbatively. Specifically, our analysis shows that it would be necessary to include at least 4 counterterms for the singlet 1S_0 and 6 counterterms for the triplet $^3S_1 - ^3D_1$ channel at NNLO. This proliferation of counterterms is expected to occur also in other partial waves because the singularity of the potential dominates over the centrifugal barrier at short distances. In the 1S_0 channel, we have seen that adding more counterterms in fact worsens the results for the effective range expansion parameters. In contrast, our non-perturbative calculations are based on just 1 and 3 counterterms respectively. The good quality of our results suggests that our choice of less counterterms cannot be refuted on the basis of phenomenology. In the deuteron case we have made a calculation to NNLO in perturbation theory. Our analysis shows that such a perturbative approach only makes sense if a finite cut-off is introduced. In any case, the cut-off theory has less predictive power, does not provide a better phenomenological description of the deuteron than our non-perturbative renormalized results and is non-convergent since NNLO corrections are numerically *much larger* (two or three orders of magnitude) than NLO ones, despite being parametrically small. In our view this is a perturbative mani-

festation of the short distance dominance which has been unveiled non-perturbatively. In addition, the difficulties faced by a perturbative treatment are simply absent in the non-perturbative approach.

One of the main goals of nuclear physics is the determination of the nucleon-nucleon interaction. From a theoretical viewpoint the disentanglement of such an interaction in terms of pion exchanges based on chiral symmetry requires dealing with non-trivial and, to some extent, unconventional non-perturbative renormalization issues in the continuum, but it is crucial because it shows our quantitative understanding of the underlying theory of quarks and gluons in the chirally symmetric broken phase. Our results also show that the singular chiral Van der Waals forces are not necessarily spurious and inconvenient features of the chiral potential. Instead, as we have shown, the singularities alone in conjunction with renormalization ideas explain much of the observed S -waves phase shifts with natural values of the chiral constants, and provide an appealing physical picture. In this regard, it is interesting to realize that based on the analogy with molecular systems, which also exhibit a long range Van der Waals force, the liquid drop model was formulated more than 60 years ago. Chiral dynamics may provide not only a closer analogy and perhaps more quantitative insights into the hydrodynamical and thermodynamical properties of nuclei but also a theoretical justification from the underlying theory of strong interactions.

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APPENDIX A: THE DEUTERON IN OPE-DISTORTED PERTURBATION THEORY WITH A CUT-OFF TO NNLO

In this appendix we illustrate the situation discussed in Sect. IX by solving numerically the set of perturbative equations (127). As we have mentioned such a calculation makes only sense within a finite cut-off scheme.

In practice, we integrate from large distances ($\sim 25\text{fm}$) with the conditions specified by Eq. (141) with some prescribed values of c_S and c_D ²⁰. This can be advanta-

geously done using the superposition principle of boundary conditions, Eq. (60), yielding in perturbation theory

$$\begin{aligned} u_\gamma(r) &= u_\gamma^{(0)}(r) + u_\gamma^{(2)}(r) + u_\gamma^{(3)}(r) + \dots \\ w_\gamma(r) &= u_\gamma^{(0)}(r) + u_\gamma^{(2)}(r) + u_\gamma^{(3)}(r) + \dots \end{aligned} \quad (\text{A1})$$

At LO the wave function can be written as

$$\begin{aligned} u_\gamma^{(0)}(r) &= u_S^{(0)}(r) + \eta^{(0)} u_D^{(0)}(r) \\ w_\gamma^{(0)}(r) &= w_S^{(0)}(r) + \eta^{(0)} w_D^{(0)}(r) \end{aligned} \quad (\text{A2})$$

and $\eta^{(0)}$ is determined from the regularity condition at the origin [41]. At LO the normalization factor is

$$\frac{1}{(A_S^{(0)})^2} = \int_0^\infty dr (u_\gamma^{(0)}(r)^2 + w_\gamma^{(0)}(r)^2) \quad (\text{A3})$$

The NLO and NNLO contributions are

$$\begin{aligned} u_\gamma^{(2)}(r) &= c_S^{(2)} u_S^{(2)}(r) + \eta^{(0)} c_D^{(2)} u_D^{(2)}(r) \\ w_\gamma^{(2)}(r) &= c_S^{(2)} w_S^{(2)}(r) + \eta^{(0)} c_D^{(2)} w_D^{(2)}(r) \\ u_\gamma^{(3)}(r) &= c_S^{(3)} u_S^{(3)}(r) + \eta^{(0)} c_D^{(3)} u_D^{(3)}(r) \\ w_\gamma^{(3)}(r) &= c_S^{(3)} w_S^{(3)}(r) + \eta^{(0)} c_D^{(3)} w_D^{(3)}(r) \end{aligned} \quad (\text{A4})$$

The advantage is that the functions appearing here only depend on the potential and the deuteron binding energy, whereas the coefficients must be determined by some additional conditions. In the first place, normalization to NLO and NNLO requires orthogonality of the wave functions to the LO solution,

$$\begin{aligned} 0 &= \int_0^\infty dr (u^{(0)}(r) u^{(2)}(r) + w^{(0)}(r) w^{(2)}(r)) \\ 0 &= \int_0^\infty dr (u^{(0)}(r) u^{(3)}(r) + w^{(0)}(r) w^{(3)}(r)) \end{aligned} \quad (\text{A5})$$

This implies the couple of linear relations²¹

$$\begin{aligned} -\eta^{(0)} \frac{c_D^{(2)}}{c_S^{(2)}} &= \frac{\int_0^\infty dr (u^{(0)}(r) u_S^{(2)}(r) + w^{(0)}(r) w_S^{(2)}(r))}{\int_0^\infty dr (u^{(0)}(r) u_D^{(2)}(r) + w^{(0)}(r) w_D^{(2)}(r))} \\ -\eta^{(0)} \frac{c_D^{(3)}}{c_S^{(3)}} &= \frac{\int_0^\infty dr (u^{(0)}(r) u_S^{(3)}(r) + w^{(0)}(r) w_S^{(3)}(r))}{\int_0^\infty dr (u^{(0)}(r) u_D^{(3)}(r) + w^{(0)}(r) w_D^{(3)}(r))} \end{aligned} \quad (\text{A6})$$

Further relations can be obtained by imposing renormalization conditions. Note that the required number of conditions increases with the order. This is similar in spirit to the procedure of adding more counterterms for the

direct use of the explicit expressions Eq. (143) involving determinants.

²¹ For instance, at $r_c = 0.5\text{fm}$ we get $c_D^{(2)} = -5.865c_S^{(2)}$ and $c_D^{(3)} = -5.545c_S^{(3)}$

²⁰ This is numerically more efficient and stable procedure than a

scattering problem discussed in Sect. IX. For instance, using the perturbative expansion for A_S and A_D

$$\begin{aligned} A_S &= A_S^{(0)} \left(1 + c_S^{(2)} + c_S^{(3)} + \dots \right) \\ A_D &= A_D^{(0)} \eta^{(0)} \left(1 + c_D^{(2)} + c_D^{(3)} + \dots \right) = \eta A_S \end{aligned} \quad (\text{A7})$$

In practice, we use a short distance cut-off r_c for the NLO and NNLO contributions only. Deuteron properties can be written to NNLO as follows

$$\begin{aligned} r_m &= r_m^{(0)} + c_S^{(2)} r_m^{(2,S)} + \eta^{(0)} c_D^{(2)} r_m^{(2,D)} \\ &\quad + c_S^{(3)} r_m^{(3,S)} + \eta^{(0)} c_D^{(3)} r_m^{(3,D)} + \dots \end{aligned} \quad (\text{A8})$$

$$\begin{aligned} Q_d &= Q_d^{(0)} + c_S^{(2)} Q_d^{(2,S)} + \eta^{(0)} c_D^{(2)} Q_d^{(2,D)} \\ &\quad + c_S^{(3)} Q_d^{(3,S)} + \eta^{(0)} c_D^{(3)} Q_d^{(3,D)} + \dots \end{aligned} \quad (\text{A9})$$

where the potential contributions have explicitly been factored out. The numerical solution requires some care, due to the short distance instabilities and oscillations. This requires using an adaptive grid to optimize the convergence. Since solutions of different orders must be mixed in the evaluation of the orthogonality conditions, Eq. (A6), and observables, Eq. (A9), we solve all LO, NLO and NNLO equations simultaneously to provide all functions on the same grid.

At NLO and fixing $r = r_c$ we demand the experimental value of η , from Eq. (142) and Eq. (A6). This way, a solution which we denote by $(c_S^{(2)}|_{\text{NLO}}, c_D^{(2)}|_{\text{NLO}})$ can be obtained. From there we can obtain deuteron properties to NLO, as a function of the perturbative cut-off r_c . In Fig. 19 we show the dependence of A_S , r_m , Q_d and p_d on r_c . As we see the NLO correction is tiny and stable

for $r_c > 0.2\text{fm}$. At NNLO we fix η and A_S . The solution is now $(c_S^{(2)}|_{\text{NNLO}}, c_D^{(2)}|_{\text{NNLO}})$ and $(c_S^{(3)}|_{\text{NNLO}}, c_D^{(3)}|_{\text{NNLO}})$. Note that in general the NLO coefficients $c_S^{(2)}$ and $c_D^{(2)}$ must be readjusted. In this case the correction is much larger than the NLO case (see Fig. 19) and the cut-off dependence is stronger due to the $1/r^6$ singularity of the NNLO potential. As a curiosity, we mention that at short distances worry-some negative D-wave probabilities show up below $r_c = 0.17\text{fm}$ at NNLO, a spurious feature which can only take place in perturbation theory and sets a unitarity bound on the short distance perturbative cut-off. Numerical results are provided in Table VIII for Set IV. Typically, we find that results do not depend dramatically on the chosen chiral couplings. We take $r_c = 0.5\text{fm}$ as a standard choice. As we see, finite cut-off perturbation theory does not work better than our non-perturbative results of Sect. V, and in fact requires one more counterterm. Actually, this is a perturbative indication that NNLO is more important than NLO, casting doubts on the convergence of the approach.

Finally, we have checked that taking the LO to be the *full* OPE potential, $\mathbf{U}_{1\pi}$ and the perturbation to be $\mathbf{U}_{2\pi}^{(2)} + \mathbf{U}_{2\pi}^{(3)}$ as a whole and keeping the cut-off $r_c > 0.1\text{fm}$ does not change the results significantly. Actually, the perturbative result does not account for the value obtained non-perturbatively, despite the apparent linear behaviour observed when changing numerically the scaling parameter λ in the region $\lambda \gg 0.1$ (see Fig. 17). This supports our conclusion that perturbation theory does not compute the slope of $A_S(\lambda)$ at the origin. In addition, even if we disregard the divergence by introducing a cut-off, the perturbative calculation does not account for the non-perturbative renormalized result.

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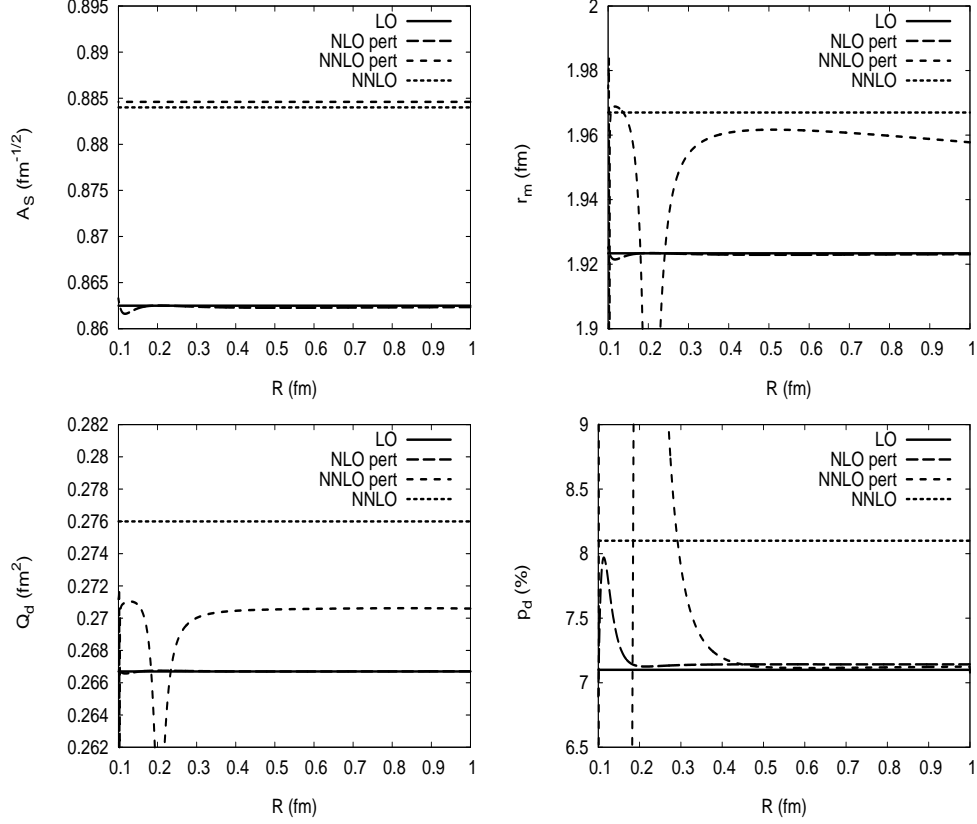


FIG. 19: Dependence of the s-wave function normalization $A_S(r_c)$, the matter radius $r_m(r_c)$, quadrupole moment $Q_d(r_c)$ and the D-wave probability $P_D(r_c)$ in the deuteron in perturbation theory to NLO and NNLO on the short distance cut-off. We use Set IV for the NNLO. In any case, the first order solution is fixed to be normalized and to reproduce the asymptotic D/S ratio, η . We compare with the experimental value and the OPE result.

TABLE VIII: Comparison of finite cut-off perturbation theory, $r_c = 0.5\text{fm}$, with renormalized non-perturbative results for the Deuteron properties. We use the non-relativistic relation $\gamma = \sqrt{2\mu_{np}B}$ with $B = 2.224575(9)$. The errors quoted in the perturbative calculations reflect the uncertainty in the input parameters γ , η and A_S . Similarly, the errors quoted in the TPE reflect the uncertainty in the non-potential parameters γ and η . We use Set IV of low energy constants c_1, c_3 and c_4 .

	r_c	$\gamma(\text{fm}^{-1})$	η	$A_S(\text{fm}^{-1/2})$	$r_m(\text{fm})$	$Q_d(\text{fm}^2)$	P_D	$\langle r^{-1} \rangle (\text{fm}^{-1})$
Non-perturbative								
$U_{1\pi}$	0	Input	0.02633	0.8681(1)	1.9351(5)	0.2762(1)	7.31(1)%	0.476(3)
$U_{1\pi} + U_{2\pi}$	0	Input	Input	0.884(4)	1.967(6)	0.276(3)	8(1)%	0.447(5)
Perturbative								
$U_{1\pi}^{(0)}$	0	Input	0.02555	0.8625(2)	1.9233(5)	0.2667(1)	7.14(1)%	0.484(3)
$U_{1\pi}^{(2)} + U_{2\pi}^{(2)}$	0.5 fm	Input	Input	0.862(2)	1.923(4)	0.2667(3)	7.14(1)%	0.484(3)
$U_{1\pi}^{(3)} + U_{2\pi}^{(3)}$	0.5 fm	Input	Input	Input	1.962(2)	0.2705(2)	7.12(1)%	0.484(3)
Potentials								
NijmII	—	0.231605	0.02521	0.8845(8)	1.9675	0.2707	5.635%	0.4502
Reid93	—	0.231605	0.02514	0.8845(8)	1.9686	0.2703	5.699%	0.4515
Exp.	—	0.231605	0.0256(4)	0.8846(9)	1.971(6)	0.2859(3)	—	—

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